®-Compact Sets and Operators

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ملخص البحث

افرض ان E فراغ محدب محليا ،تسمى الفئة المحدودة $D \subset E$ بشبه محكمة اذا وفقط اذا $\underset{n}{Lim}(\delta_n(D,U)) = 0$ لاى جوار U للصفر فى E . قمنا فى هذا العمل بدراسة بعض انواع االفئات شبه المحكمة (تسمى محكمة–(ر)) التى متتابعاتها من الاقطار النونية تتقارب الى الصفر بمعدلات مختلفة، وبرهنا انه اذا كان $E_i, i = 1,2,...,n$ فئات محكمة–(ر) فأن االفئة $\prod_{i=1}^{n} E_i$ تكون محكمة–(ر).

Abstract

For any locally convex space E, the bounded subset D of E is Precompact if and only if $\underset{n}{Lim}(\delta_n(D,U)) = 0$ for any neighborhood of zero in E. In this work we study some types of Precompact sets (called ®-compact) whose sequences of n-diameters converge to zero in different rates(rapidly, radically,...), and we prove that if E_i, $i = 1,2,...,n$, are ®-compact sets, then $\prod_{i=1}^{n} E_i$ is ®-compact.

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1. Introduction and Preliminaries.

In [3] Farid and Ramadan studied some types of compact sets (which called ®-compact) in normed spaces whose sequences of n-diameters converge to zero in different rates. Also they proved that n-diameters of finite cartesian products of ®-compact sets is ®-compact. In this work we are interested in studying ®-compact sets in locally convex spaces and ®-compact operators between locally convex spaces.

By c_0 we denote the space of all sequences of real numbers that converge to zero. By (S), we denote the space of all rapidly decreasing sequences of real numbers given by:

$$(S)=\left\{(\lambda_n)_{n=1}^{\infty} : \sup_n n^{\alpha}|\lambda_n|<\infty \ \forall \alpha>0\right\}.$$

By $\Lambda(\alpha)$, $\alpha=(\alpha_n)_{n=1}^{\infty}$, $0<\alpha_1\leq\alpha_2\leq\ldots$ we denote the power series spaces of all sequences of real numbers given by:

$$\Lambda(\alpha)=\left\{(\lambda_n)_{n=1}^{\infty} : \sup_n R^{\alpha_n}|\lambda_n|<\infty \ \forall R>0\right\}.$$

By (R), we denote the space of all radical sequences of real numbers given by:

$$(R)=\left\{(\lambda_n)_{n=1}^{\infty} : \lim_n \sqrt[n]{|\lambda_n|}=0\right\}$$

By $[x]$ we mean the integer part of real number x such that $[x]=\alpha$ if $x=\alpha+\beta, 0\leq\beta<1$.

All spaces considered will be locally convex spaces. By L(E,F), we denote the spaces of all linear continuous operators.

For unexplained terminology the reader is referred to [2,4,6,7,].

Remarks: From [3], proposition 1, we have

(1) For (S), $(\lambda_n)_{n=1}^{\infty} \in (S)$ if and only if $\lim_n |\lambda_n| n^{\alpha} = 0$ for all $\alpha > 0$.

(2) The space (S) is a special case of the space $\Lambda(\alpha)$.

(3) If $(\lambda_n)_{n=1}^{\infty} \in (R)$, then $(\lambda_n n^{\alpha})_{n=1}^{\infty} \in (R)$ for all $\alpha > 0$.

(4) Each radical sequence is rapidly decreasing and the converse is not necessarily true.

In fact, if we take $\lambda_n = \frac{1}{2^n}, n \in N$, then $\lim_n \lambda_n n^{\alpha} = \lim_n \frac{n^{\alpha}}{2^n} = 0$ for all $\alpha > 0$. But since $\sqrt[n]{\frac{1}{2^n}} = \frac{1}{2}$, we have $(\lambda_n)_{n=1}^{\infty} \in (S) \setminus (R)$.

Definition 1.1. [3] A sequence ideal ® on a scalar field is a subset of the space l_{∞} (the space of all bounded sequences of real numbers) satisfying the following conditions:

(i) $e_i \in$ ®, where $e_i = (0,0,\ldots,1,\ldots)$ the one in the ith place.

(ii) If $x_1, x_2 \in$ ®, then $x_1 + x_2 \in$ ®.

(iii) If $y \in l_{\infty}$ and $x \in$ ®, then $x.y \in$ ®.

(iv) If the sequence $x = (x_0, x_1, \ldots) \in$ ®, then $(x_{\left[\frac{n}{2}\right]})_{n=1}^{\infty} =$

$(x_0, x_0, x_1, x_1, \ldots) \in$ ®.

Definition 1.2. Let E be a sequence ideal. We call the operator $D: E \to E$ defined by $D((x_n)_{n=0}^{\infty}) = (x_{\left[\frac{n}{2}\right]})_{n=0}^{\infty}$ the dilatation operator and condition(iv) in definition (1.1) the dilatation property.

Note that the sequence spaces c_0, (S), (R) and $\Lambda(\alpha)$ are examples of sequence ideals (see[4], p. 9).

Definition 1.3. Let A, D be two absolutely convex sets in a topological vector space E such that D absorbs A, i.e. there exists $\lambda > 0$ such that $A \subset \rho D$ for all $\rho > \lambda$. For a subspace F of E we define:

$$\delta(A,D;F) = \inf\{r > 0 : A \subset rD + F\}.$$

the nth diameter of A with respect to D is defined as

$$\delta_n(A,D) = \inf\{\delta(A,D;F) : \dim(F) \le n\}, \quad n = 0,1,2,\ldots,$$

and it satisfies the following properties :

(1) $\delta_0(A,D) \ge \delta_1(A,D) \ge \ldots \ge \delta_n(A,D) \ldots \ge 0.$

(2) $\delta_n(A,D) = 0$ if and only if A is contained in a linear subspace of E of dimension at most n.

(3) If A is a bounded subset of E, then A is precompact if and only if

$$\lim_n(\delta_n(A,U)) = 0 \quad \forall U \in \mu(E),$$

where $\mu(E)$ is a local base of zero in E.

(4) If $T : E \to F$ is a linear operator , then

$$\delta_n(T(A),T(D)) \le \delta_n(A,D).$$

(5) If $A_1 \subset A$ and $D \subset D_1$, then $\delta_n(A_1,D_1) \le \delta_n(A,D)$.

(6) Let $T : E \to F$ and $S : F \to G$ be linear operators and let U, V and W be absolutely convex sets in E, F and G respectively, such that V absorbs T(U) and W absorbs S(V).Then

$$\delta_{n+m}(ST(U),W) \le \delta_n(T(U),V)\delta_m(S(V),W).$$

2. ®-Compact Sets

Definition 2.1. For a sequence ideal $® \subset c_0$, a subset D of a locally convex space E is called ®-compact if

$(\delta_n(D,U)_{n=0}^{\infty}) \in ®$ for all $U \in \mu(E)$.

For examples:

(1) Every finite set is ®-compact.

(2) If

$$D = \left\{(x_n) : \sum_{n=1}^{\infty} |x_n| 2^n \le 1\right\} \text{ and } B = \left\{(x_n) : \sum_{n=1}^{\infty} |x_n| n \le 1\right\}$$

are subsets of l_1, then according to [7], 9.1.3 we have $\delta_n(D, \varepsilon B_{l_1}) = \frac{1}{2^n \varepsilon}$ and $\delta_n(B, \varepsilon B_{l_1}) = \frac{1}{n\varepsilon}$, where B_{l_1} is the closed unit ball in

$(x_{\left[\frac{n}{2}\right]})_{n=1}^{\infty} = (x_0, x_0, x_1, x_1, \ldots) \in l_1$. Hence D is precompact and rapidly-compact, but not radically compact and B is precompact but not rapidly-compact.

Let E_i, $i = 1,2,$ be a locally convex spaces whose topology are induced by an increasing sequences of seminorms $(q_k^i)_{k=1}^{\infty}$. If we denote by $U_{k,n}^i$ the set of all $x \in E_i$ for which $q_k^i(x) < \frac{1}{n}$, then the collection $\mu(E_i) = \{U_{k,n}^i : k,n \in N\}$ is a local base of zero in E_i [6].

By $(E_1 \times E_2)_\infty$ and $(E_1 \times E_2)_p$ we denote the cartesian product $E_1 \times E_2$ equipped with the following sequence of seminorms:

1) $Q_{i,j}^{\infty}(x) = Q_{i,j}^{\infty}((x_1, x_2)) = \max(q_i^1(x_1), q_j^2(x_2))$.

2) $Q_{i,j}^{p}(x) = Q_{i,j}^{p}((x_1, x_2)) = \sqrt[p]{(q_i^1(x_1))^p + (q_j^2(x_2))^p}$.

Let

$$U_{i,j,n}^{\infty} = \left\{x \in E_1 \times E_2 : Q_{i,j}^{\infty}(x) \,\pi\, \frac{1}{n}\right\},$$

$$U^{p}_{i,j,n} = \left\{ x \in E_1 \times E_2 : Q^{p}_{i,j}(x) \pi \frac{1}{n} \right\}.$$

Proposition 2.2. For all $i, j, n, m \in N$ we have

1) $U^{\infty}_{i,j,n} = U^{1}_{i,n} \times U^{2}_{j,n}$.

2) $U^{\infty}_{i,j,\max(n,m)} \subset U^{1}_{i,n} \times U^{2}_{j,m} \subset U^{\infty}_{i,j,\min(n,m)}$.

3) $\frac{1}{\sqrt[p]{2}} U^{1}_{i,n} \times U^{2}_{j,n} \subset U^{p}_{i,j,n} \subset U^{1}_{i,n} \times U^{2}_{j,n}$.

Proof.

1) Since $x = (x_1, x_2) \in U^{\infty}_{i,j,n}$ if and only if $q^{1}_{i}(x_1) < \frac{1}{n}$ and $q^{2}_{j}(x_2) < \frac{1}{n}$, then $U^{\infty}_{i,j,n} = U^{1}_{i,n} \times U^{2}_{j,n}$.

2) From (1) it follows that $U^{\infty}_{i,j,\max(n,m)} = U^{1}_{i,\max(n,m)} \times U^{2}_{j,\max(n,m)} \subset U^{1}_{i,n} \times U^{2}_{j,m} \subset U^{1}_{i,\min(n,m)} \times U^{2}_{j,\min(n,m)} = U^{\infty}_{i,j,\min(n,m)}$.

3) Similarly like part (1) we can show that

$$\frac{1}{\sqrt[p]{2}} U^{1}_{i,n} \times U^{2}_{j,n} \subset U^{p}_{i,j,n} \subset U^{1}_{i,n} \times U^{2}_{j,n}.$$

Remark: Proposition (2.2) gives the following results:

1) $\mu((E_1 \times E_2)_{\infty}) = \{U^{\infty}_{i,j,n} : i,j,n \in N\}$ is a local base of zero in $(E_1 \times E_2)_{\infty}$.

2) $\mu((E_1 \times E_2)_{p}) = \{U^{p}_{i,j,n} : i,j,n \in N\}$ is a local base of zero in $(E_1 \times E_2)_{p}$.

Proposition 2.3. If B_1 and B_2 are two bounded subsets of locally convex spaces E_1 and E_2 respectively, then

1) $\delta^{\infty}_{s+m}(B_1 \times B_2, U^{\infty}_{i,j,n}) \leq \max(\delta_s(B_1, U^1_{i,n}), \delta_m(B_2, U^2_{j,n}))$.

2) $\delta^{p}_{s+m}(B_1 \times B_2, U^{p}_{i,j,n}) \leq \sqrt[p]{2}\ \max(\delta_s(B_1, U^1_{i,n}), \delta_m(B_2, U^2_{j,n}))$.

Proof.

1) Let $\varepsilon > 0$, from definition (1.4) there exist subspaces $F_1 \subset E_1$ and $F_2 \subset E_2$ of dimention at most s and m respectively such that

$$B_1 \subset (\delta_s(B_1, U^1_{i,n}) + \varepsilon)\, U^1_{i,n} + F_1,$$

$$B_2 \subset (\delta_m(B_2, U^2_{j,n}) + \varepsilon)\, U^2_{j,n} + F_2.$$

Consequently,

$B_1 \times B_2 \subset$

$$\max(\delta_s(B_1, U^1_{i,n}) + \varepsilon, \delta_m(B_2, U^2_{j,n}) + \varepsilon)\, U^1_{i,n} \times U^2_{j,n} + F_1 \times F_2 =$$

$$\max(\delta_s(B_1, U^1_{i,n}) + \varepsilon,\ \delta_m(B_2, U^2_{j,n}) + \varepsilon)\, U^{\infty}_{i,j,n} + F_1 \times F_2.$$

Since $\dim(F_1 \times F_2) < s + m$, then

$$\delta^{\infty}_{s+m}(B_1 \times B_2, U^{\infty}_{i,j,n}) \leq \max(\delta_s(B_1, U^1_{i,n}),\ \delta_m(B_2, U^2_{j,n})) + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we have

$$\delta^{\infty}_{s+m}(B_1 \times B_2, U^{\infty}_{i,j,n}) \leq \max(\delta_s(B_1, U^1_{i,n}),\ \delta_m(B_2, U^2_{j,n})).$$

2) Similarly for $\delta^{p}_{s+m}(B_1 \times B_2, U^{p}_{i,j,n})$ we get

$$\delta^{p}_{s+m}(B_1 \times B_2, U^{p}_{i,j,n}) \leq \sqrt[p]{2}\ \max(\delta_s(B_1, U^1_{i,n}),\ \delta_m(B_2, U^2_{j,n})).$$

Corollary 2.4.

1) $\delta_s^{\infty}(B_1 \times B_2, U_{i,j,n}^{\infty}) \le \max(\delta_{\left[\frac{s}{2}\right]}(B_1, U_{i,n}^1), \delta_{\left[\frac{s}{2}\right]}(B_2, U_{j,n}^2))$.

2) $\delta_s^{p}(B_1 \times B_2, U_{i,j,n}^{p}) \le \sqrt[p]{2}\ \max(\delta_{\left[\frac{s}{2}\right]}(B_1, U_{i,n}^1), \delta_{\left[\frac{s}{2}\right]}(B_2, U_{j,n}^2))$.

With the same argument, we can generalize proposition (2.3) to a finite cartesian Product as follows:

Proposition 2.5. For fixed i=1,…,k, let B_i, be any bounded subset of a locally convex space E_i. Then

1) $\delta_s^{\infty}(\prod_{i=1}^{k} B_i, U_{m_1,\dots m_k,n}^{\infty}) \le$

$$\min\left\{\max(\delta_{n_1}(B_1, U_{m_1,n}^1), \dots, \delta_{n_k}(B_k, U_{m_k,n}^k)) : \sum_{i=1}^{k} n_i \le s\right\}.$$

2) $\delta_s^{p}(\prod_{i=1}^{k} B_i, U_{m_1,\dots m_k,n}^{p}) \le$

$$\sqrt[p]{2}\ \min\left\{\max(\delta_{n_1}(B_1, U_{m_1,n}^1), \dots, \delta_{n_k}(B_k, U_{m_k,n}^k)) : \sum_{i=1}^{k} n_i \le s\right\}.$$

(The minimum is taken over all choices of $n_1 + n_2 + \dots + n_k \le s$).

Proof. From the proof of proposition (2.3) we get

$B_i \subset (\delta_{n_i}(B_i, U_{m_i,n}^i) + \varepsilon) U_{m_i,n}^i + F_i \quad \forall i = 1,2,\dots,k$, therefore

$\prod_{i=1}^{k} B_i \subset$

$$(\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n}))+\varepsilon)\prod_{i=1}^{k}U^i_{m_i,n}+F_1\times\dots\times F_k=$$

$$(\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n}))+\varepsilon)U^\infty_{m_1,\dots,m_k,n}+F_1\times\dots\times F_k.$$

Since $\dim(F_1\times\dots\times F_k)=\sum_{i=1}^{n}\dim(F_i)\le n_1+\dots+n_k\le s$, then

$$\delta^\infty_s(\prod_{i=1}^{k}B_i,U^\infty_{m_1,\dots m_k,n})\le(\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n}))+\varepsilon),$$

and since $\varepsilon>0$ is arbitrary, we have

$$\delta^\infty_s(\prod_{i=1}^{k}B_i,U^\infty_{m_1,\dots m_k,n})\le\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n})).$$

Since this estimation is true for any choice of $n_1+n_2+\dots+n_k\le s$, then

$$\delta^\infty_s(\prod_{i=1}^{k}B_i,U^\infty_{m_1,\dots m_k,n})\le$$

$$\min\left\{\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n})):\sum_{i=1}^{k}n_i\le s\right\}.$$

Similarly for $\delta^p_s(\prod_{i=1}^{k}B_i,U^p_{m_1,\dots m_k,n})$, we have

$$\delta^p_s(\prod_{i=1}^{k}B_i,U^p_{m_1,\dots m_k,n})\le$$

$$\sqrt[p]{2}\,\min\left\{\max(\delta_{n_1}(B_1,U^1_{m_1,n}),\dots,\delta_{n_k}(B_k,U^k_{m_k,n})):\sum_{i=1}^{k}n_i\le s\right\}.$$

Theorem 2.6. The cartiseain product of two ®-compact sets is ®-compact .

Proof. Let B_1 and B_2 be any two ®- compact subsets of locally convex spaces E_1 and E_2 respectively. From the definition of ®-compact set, we have

$$(\delta_s(B_1,U^1_{i,n}))_{s=0}^{\infty} \in ® \text{ and } (\delta_s(B_2,U^2_{j,n}))_{s=0}^{\infty} \in ® \ \forall i,j,n \in N,$$

hence

$$(\delta_{\left[\frac{s}{2}\right]}(B_1,U^1_{i,n}))_{s=0}^{\infty} \in ® \text{ and } (\delta_{\left[\frac{s}{2}\right]}(B_2,U^2_{j,n}))_{s=0}^{\infty} \in ® \ \forall i,j,n \in N.$$

But since

$$\delta_s^{\infty}(\ B_1 \times B_2,\ U^{\infty}_{i,j,n}) \leq \ \max(\delta_{\left[\frac{s}{2}\right]}(B_1,U^1_{i,n}),\ \delta_{\left[\frac{s}{2}\right]}(B_2,U^2_{j,n})) \leq$$

$$\delta_{\left[\frac{s}{2}\right]}(B_1,U^1_{i,n}) + \delta_{\left[\frac{s}{2}\right]}(B_2,U^2_{j,n}) \ \forall i,j,s,n \in N,$$

then

$$(\delta_s^{\infty}(\ B_1 \times B_2,\ U^{\infty}_{i,j,n}))_{s=0}^{\infty} \in ® \ \forall i,j,n \in N,$$

and therefore $B_1 \times B_2$ is ®-compact subset of $(E_1 \times E_2)_{\infty}$.

With the same argument, we can generalize theorem (2.6) to finite cartesian product of ®-compact sets.

Theorem 2.7. The continuous linear image of any ®-compact set is ®-compact.

Proof. Let E and F be two locally convex spaces. Suppose T is any linear continuous operator. If B is ®-compact subset of E , then

$$(\delta_n(\ B,U\ ))_{n=0}^{\infty} \in ® \quad \forall U \in \mu(E).$$

Since T is continuous, then for all $W \in \mu(F)$ there exist a neighborhood $U \in \mu(E)$ such that $T(U) \subset W$. So

$$\delta_n(T(B),W) \le \delta_n(T(B)),T(U) \le \delta_n(B,U) \quad \forall n \in N.$$

It follows that $(\delta_n(T(B),W))_{n=0}^{\infty} \in$®, hence T(B) is ®-compact subset of F.

Theorem 2.8. Let B_1 and B_2 be any two subsets of locally convex spaces E_1 and E_2 respectively. If $B_1 \times B_2$ is ®-compact subset of $E_1 \times E_2$, then B_1 and B_2 are ®-compact sets.

Proof. Since the projection p_i from $E_1 \times E_2$ into E_i is continuous and $P_i(B_1 \times B_2) = B_i$, we conclude that B_i is ®-compact subset of E_i for all $i = 1,2$.

3. ®-Compact Operators.

Definition 3.1. For a sequence ideal ® $\subset c_0$ and two locally convex spaces E and $F, T \in L(E,F)$ is called an ®- compact operator if there exists a neighborhood V of zero in E such that $(\delta_n(T(V),U))_{n=0}^{\infty} \in$ ® for all $U \in \mu(F)$.

Proposition 3.2. If $T: E \to F$ is ®-compact operator , and if A is a subspace of E, then the restricted operator $T|A : A \to F$ is an ®-compact operator.

Proof. Since $T: E \to F$ is ®- compact operator, there exists a neighborhood U of zero in E such that $T(U)$ is ®-compact subset of F, and if $U_o = U \mathrm{I} A \subset U$, then $T|A(U_0) = T(U_0) \subset T(U)$. Since U_0 is a neighborhood of zero in A, then $T|A$ is ®- compact operator.

Proposition 3.3. Let $T \in L(E,F), S \in L(F,G)$ and $H \in L(K,E)$. If

T is ®- compact operator, then $S o T$ and $T o H$ are ®-compact operators.

Proof. Since T is ®- compact operator, there exists a neighborhood V' of zero in E such that

$$(\delta_r(T(V'),W'))_{r=0}^{\infty} \in ® \ \forall W' \in \mu(F).$$

Hence

$$(\delta_{\left[\frac{r}{2}\right]}(T(V'),W'))_{r=0}^{\infty} \in ® \ \forall W' \in \mu(F).$$

Now for $W \in \mu(G)$, and if we let $V = S^{-1}(W)$, then by (1.3.6) we have

$$\delta_r(ST(V'),W) \leq \delta_{\left[\frac{r}{2}\right]}(T(V'),V).\delta_{\left[\frac{r}{2}\right]}(S(V),W),\ r \in N.$$

Also for $W' \in \mu(F)$, and if we let $U = H^{-1}(V')$, then

$$\delta_r(TH(U),W') \leq \delta_{\left[\frac{r}{2}\right]}(H(U),V').\delta_{\left[\frac{r}{2}\right]}(T(V),W'),\ r \in N.$$

Since $(\delta_{\left[\frac{r}{2}\right]}(H(U),V'))_{r=0}^{\infty}$, $(\delta_{\left[\frac{r}{2}\right]}(S(V),W))_{r=0}^{\infty} \in l_{\infty}$ and

$(\delta_{\left[\frac{r}{2}\right]}(T(V'),W'))_{r=0}^{\infty} \in ®$ for all $W \in \mu(G)$, $W' \in \mu(F)$, we conclude

that $S o T$ and $T o H$ are ®- compact operators.

Proposition 3.4. If $S_1, S_2 \in L(E,F)$ are ®- compact operators, then $S_1 + S_2$ is ®- compact operator.

Proof. Since S_1 and S_2 are ®- compact operators, there exist neighborhood's V_1 and V_2 of zero in E such that

$$(\delta_{\left[\frac{n}{2}\right]}(S_1(V_1),U))_{n=0}^{\infty} \in ® \text{ and } (\delta_{\left[\frac{n}{2}\right]}(S_2(V_2),U))_{n=0}^{\infty} \in ®$$
$$\forall U \in \mu(F).$$

If $W \in \mu(F)$, there exists a neighborhood U of zero in F such that $U+U \subset W$. According to the definitions of $\delta_n(S_1(V_1),U)$ and $\delta_m(S_2(V_2),U)$ we have for any $\varepsilon > 0$ there exist subspaces F_1 and F_2 of F with $\dim(F_1) \le n$ and $\dim(F_2) \le m$ such that

$$S_1(V_1) \subset (\delta_n(S_1(V_1),U)+\varepsilon)U + F_1,$$

$$S_2(V_2) \subset (\delta_m(S_2(V_2),U)+\varepsilon)U + F_2.$$

Hence we have

$$S_1(V_1)+S_2(V_2) \subset (\max(\delta_n(S_1(V_1),U),\delta_m(S_2(V_2),U))+\varepsilon)(U+U)+(F_1+F_2) \subset$$

$$(\max(\delta_n(S_1(V_1),U),\delta_m(S_2(V_2),U))+\varepsilon)W+(F_1+F_2).$$

Since $\dim(F_1+F_2) \le n+m$, then

$$\delta_{n+m}(S_1(V_1)+S_2(V_2),W) \le (\max(\delta_n(S_1(V_1),U),\delta_m(S_2(V_2),U))+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\delta_{n+m}(S_1(V_1)+S_2(V_2),W) \le (\max(\delta_n(S_1(V_1),U),\delta_m(S_2(V_2),U)).$$

Now if we let $V = V_1 \cap V_2$, then $(S_1+S_2)(V) \subset S_1(V_1)+S_2(V_2)$.

It follows that

$$\delta_{n+m}((S_1+S_2)(V),W) \le \delta_{n+m}(S_1(V_1)+S_2(V_2),W) \le$$
$$\max(\delta_n(S_1(V_1),U),\delta_m(S_2(V_2),U)).$$

Hence for all $W \in \mu(F)$, we have

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