Embedding and hyperplane of point-line geometry of type $D_{n,k}(F)$

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ملخص البحث

أثبت Shult [7] لفضاء Grasmann المنتهي القابل للطمر أن كل مستوى فوقي يأتي من الطمر. في هذا البحث نبرهن أن هندسة نقطة–خط من نوع $D_{n,k}$ حيث $k\geq2$ $n\geq k+3$ ، قابلة للطمر في فضاء إسقاطي وأن كل مستوى فوقي عادي يأتي من ذلك الطمر.

Abstract

Shult [7] proved that every hyperplane of embeddable Grassmann space of finite rank, arises from an embedding. In this paper it is proved that the point-line geometry of type $D_{n,k}$ ($k\geq2$, $n\geq k+3$) can be embedded in projective space and every regular hyperplane arises from an embedding using the result in [7].

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Introduction

Many authors interested in embeddability of some geometeries in a projective space. Lynn Batten [4] found an intimate relationship between affine and projective planes by showing that any affine plane is embeddable in projective plane. Cooperstein and Shult [3] showed that any geometric hyperplane arises from an embedding for the following Lie-incidence geometries: $A_{n,2}$, $D_{5,5}$ and $E_{6,1}$. Agreat effort has been spent by Shult [8] showing that the half-spin geometry of type $D_{n,n}$ is embeddable in a projective space. Ronan Theory [5] presented a basic tool for certain strong parapolar spaces to show that whether the hyperplanes arise from an embedding. It has been proved in [9] the embeddability of certain class of geometries $D_{n,2}$, $D_{n,2}$ and $D_{n,3}$ In this paper we use the result in [7] to prove the embeddability of the point-line geometry $D_{n,k}$, $k \geq 2$ and $n \geq k+3$.

A subspace of a point-line geometry $\Gamma=(P, L)$ is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. $\langle X \rangle$ means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called ***thick*** lines, those incident with exactly two points are called ***thin*** lines.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there exist a chain of distinct subspaces $\varphi \neq X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$. For example rank(φ)=-1, rank($\{p\}$)=0 where p is a point and rank(l)=1 where l a line.

In a point-line geometry Γ=(P, L), a path of length n is a sequence of n+1 $(x_0, x_1, .., x_n)$ where, (x_i, x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. ***A geodesic*** from a point x to a point y is a path of minimal possible length with initial point x and end point y. We denote this length by $d_\Gamma(x, y)$.

A geometry Γ is called ***connected*** if and only if for any two of its points there is a path connecting them. A subset X of P is said to be ***convex*** if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry Γ=(P, L) satisfying the axiom:

For each point-line pair (p, l) with p not incident with l; p is collinear with one or all points of l, that is $|p^{\perp} \cap l|=1$ or else $p^{\perp} \supset l$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of Γ for every point $p \in$ P.

A point-line geometry Γ=(P, L) is called ***a projective plane*** if and only if it satisfies the following conditions [5]:

(i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of them are on a line.

A point-line geometry Γ=(P, L) is called ***a projective space*** if the following conditions are satisfied:

(i) every two points lie exactly on one line ,

(ii) if l_1, l_2 are two lines $l_1 \cap l_2 \neq \emptyset$, then $\langle l_1, l_2 \rangle$ is a projective plane. ($\langle l_1, l_2 \rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

A point-line geometry Γ=(P, L) is called ***a parapolar space*** if and only if it satisfies the following properties:

i- Γ is a connected gamma space,

ii- For every line l; $l^{\perp}$ s not a singular subspace,

iii- For every pair of non-collinear points x, y; $x^{\perp} \cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If x, y are distinct points in P, and if $|x^{\perp} \cap y^{\perp}|=1$, then (x, y) is called *a special pair*, and if $x^{\perp} \cap y^{\perp}$ is a polar space, then (x, y) is called *a polar pair* (or *a symplectic pair*). A parapolar space is called ***a strong parapolar*** space if it has no special pairs.

Embeddings.[6] Let Γ=(P, L) be a point-line geometry, that is, an incidence system of points P and lines L, such that each line is viewed as a set of points. ***A projective embedding*** of a point-line geometry Γ into the projective space $\mathbb{P}$(V) of all proper subspaces of the vector space V is an injective mapping e: P→ projective points of $\mathbb{P}$(V)=1-spaces of V such that

i- e(l) is a projective line for each l of L, and

ii- the image points e(P) span $\mathbb{P}$(V).

Such an embedding is denoted by the symbol e: Γ→ $\mathbb{P}$(V).

A geometric hyperplane of a point-line geometry is a proper subspace which meets each line non-trivially. Let e: Γ→ $\mathbb{P}$(V) be a projective embedding of the point line geometry Γ=(P,L). Suppose $\mathbb{H}$ is a projective hyperplane of $\mathbb{P}$(V), the set $H(\mathbb{H}):=\{x\in P \mid e(x)\in \mathbb{H}\}$ is a hyperplane of Γ. A hyperplane H of Γ is said to arise from the embedding e if and only if it has the form $H=H(\mathbb{H})$

Morphisms of embeddings. Let τ:V→W be a semilinear transfor- mation of vector spaces. This induces a partial mapping of the corresponding projective spaces $\mathbb{P}$(V) and $\mathbb{P}$(W), sending points of $\mathbb{P}$(V) not contained in kerτ in $\mathbb{P}$(V) to projective points of **P**(W). If e: $\Gamma \to$ **P**(V) is a projective embedding of the point-line geometry Γ, then composition with the partial map τ can yield a new embedding $e\tau$ if and only if

i- $\tau$is a surjective semilinear transformation, and

ii- for any point p and q of Γ, ker τ meets the subspace $\langle e(p),e(q)\rangle$ at the zero subspace of V. We call the transfer from embedding e to embedding $e\tau$ ***a morphism of embeddings*** and write $e \to e\tau$. In general if we insist that $e \to e'$ is a morphism of embedding, it means that $e' = e\tau$ for an appropriate semilinear transformation τ. If ker τ is the zero vector space, e and e' are said to be ***equivalent embeddings***.

An embedding u: $\Gamma \to$ P(V) is said to be relatively universal if and only if the existence of a morphism $w \to u$ implies w is equivalent to u.

2- Old results and Notations

Shult has proved in the following theorem that every hyperplane of embeddable Grassmannspace of finite rank, arises from an embeding. For characterization and construction of $A_{n,k}$ see [2] and [1].

2.1 Theorem [7]. *Let* Γ=(P,L) be a Grassmann space of type $A_{n,k}(F)$, where F is a field, $1<k<n$. Then every geometric

hyperplane of Γ arises from the universal embedding of Γ in P(W) where W is the k-fold wedge product of the n-space V over F with itself.

For embeddings of different Lie incidence geometries, there are three methods showing that every hyperplane arises from embedding that have been successful.

Method 1. (Circuitry) A circuit $C=(x_0,x_1,\ldots,x_{n-1})$ in the collinearity graph X=(P-H,~) is said to be *decomposable* if it is the sum of circuits C_0, $C_1,\ldots,$ in X such that each C_i lies in some symplecton S_i of the parapolar space Γ. We say that C is *minimal indecomposable* if C is not decomposable and all circuits of smaller length are decomposable.

By results of Ronan [5], for a strong parapolar space, it is sufficient to show that for any hyperplane H of Γ, that in the subgraph of (P,~) induced on P-H, every circuit C is a sum of triangles and 4-circuits.

Method 2. (Inductive construction of a functional) If e: $\Gamma \rightarrow P(V)$ is the embedding, we wish to show that for each hyperplane H of Γ, there exists a functional $h:V \rightarrow F$ of V such that $e \circ h$ vanishes on H but never vanishes on P-H. Inductively there exists a family S of subgeometries of Γ belonging to a parameterized family of geometries containing Γ and such that the restriction of e to S in S is still relatively universal. Then for each $S \in$ S, there exists by induction a functional

$$H_S : \langle e(S) \rangle := W_S \to F$$

Which vanishes on e(S-H) (if the latter is empty, of cource, h_S=0). This method was used in [7].

Method 3. (The direct sum method) This method also use induction indifferent way. We asume as before that e: $\Gamma \to P(V)$ is relatively universal. Again Γ is assumed to belong a family of geometries parameterized by an integer-valued function τ. The theorem being proved asserts that for some function $\delta : Z \to Z$, if $\dim V \geq \delta(\tau(\Gamma))$, then $\dim V = \delta(\tau(\Gamma))$ and every hyperplane arises from embeding. The geometries Γ must have the property that they always possess two subgeometries S_1 and S_2 in the same parameterized family satisfying

i- $\langle S_1, S_2 \rangle_\Gamma = P$, and

ii- $\delta(\tau(S_1)) + \delta(\tau(S_2)) \leq \delta(\tau(\Gamma))$.

Induction on the S_i immediately yields the fact that $V = \langle e(S_1) \rangle \oplus \langle e(S_2) \rangle$ and that $\dim \langle e(S_i) \rangle_V = \delta(\tau(S_i))$. By induction, the hyperplane H can be assumed to meet each S_i properly at a hyperplane H_i of S_i . One sets X to be all points x of H such that $e(x)$ is not in the subspace $U:=\langle e(H_1)\rangle \oplus \langle e(S_2)\rangle$ of codimension 2 in V. For x, y∈X, one write

$$x \sim y \text{ if and only if } U \oplus \langle x\rangle = U \oplus \langle y\rangle.$$

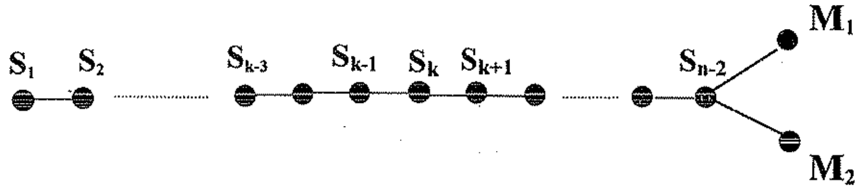
The rest of the proof consists in showing that the graph (X,~) is connected.

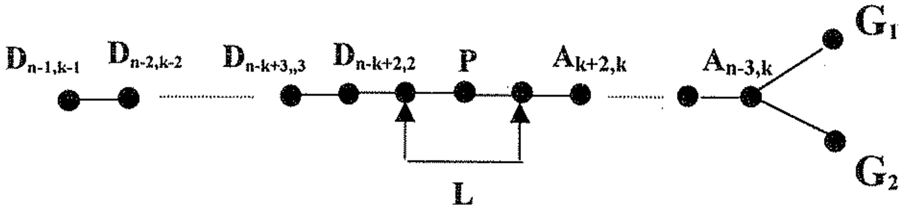
Summarizing known resuls, we have:

Geometry	Reference	Method
$E_{6,1}$, $D_{5,5}$ and $A_{n,2}$	Cooperstein and Shult [3]	Circuitry
Half-spin geometry, $D_{n,n}$	Shult [8]	Direct sum
$A_{n,k}$, $2\leq[(n+1)/2]$	Shult [7]	Functional method

We present some facts about the point-line geometry of type $D_{n,k}$ that can be found in [9] and [10].

Construction of $D_{n,k}$ (F) [10]. Consider the polar space $\Delta=\Omega^+(2n, F)$ that comes from a vector space V of dimension 2n over a finite field F=GF(q) with a symmetric hyperbolic bilinear form B. S_I is the set of all totally isotropic i-dimensional subspaces of V; $1\leq i\leq n-2$. The two classes M_1, M_2 consist of maximal totally isotropic n-dimensional subspaces. Two n-spaces fall in the same class if their intersection is of odd dimension.





$D_{n,k}(F)$

The geometry of type $\mathbf{D_{n,k}(F)}$ is the point-line geometry (P, L), whose set of points P is the collection of all k-dimensional subspaces of the vector space V, and whose lines are the pairs (A, B) where A is (k-1)-dimensional subspace of (k+1)-subspace B–that is, the set of (k-1, k+1)-subspace of V. A point C is incident with a line (A, B) if and only if $A \subset C \subset B$ as a subspaces of V.

To define the collinearity, let C_1 and C_2 be two point (the points are the T.I k-spaces), then C_1 is collinear to C_2 if and only if the intersection of $C_1 \cap C_2$=(k-1)-space and $\langle C_1, C_2\rangle$=(k+1)-space.

The elements of the classes G_1 and G_2 are Grassmannian geometries of type $A_{n-1,k}$.

There are two kinds of symlecta (1) The first kind is the convex polar spaces $A_{3,2}$ that represent the (k-2, k+2) subspaces of V. Then symplecton S of kind $A_{3,2}$ is the set of TI k-dimensional spaces that contain the TI (k-2)-dimensional space and contained in the TI (k+2)-dimensional space. (2) The second kind of symplecta is the convex polar spaces of type $D_{n-k+1,1}$ that represent the collection of all TI (k-1)-subspaces of V. Thus this kind of symplecta is defined as the collection of all TI k-subspaces of V that contain such TI (k-1)-spaces.

2.2 Proposition [9]. *Let* Γ=(P, L) *be the geometry* of type $D_{n,k}(F)$. *Thus*:

i- Γ *is of diameter* k+1,.

ii- Γ *is a weak parapolar geometry.*

3- The main result

Before proving the main theorem we need to present the definition of the regular hyperplane and we show that the point line geometry $D_{n,k}$ is a geometric subspace of the Grassmann geometry $A_{n,k}$.

3.1 Definition [9]. Let Γ be a sub-geometry of a geometry Γ'. Let H be a hyperplane of Γ. If there exist a hyperplane H' of Γ' such that $H=H'\cap\Gamma$ we say that H is a regular hyperplane of Γ with respect to Γ'. Otherwise we say that H is irregular with respect to Γ'.

For example if H' is a hyperplane of Γ', then $K=H'\cap\Gamma$ is a hyperplane of Γ implies that K is regular with respect to Γ'.

3.2. Lemma $\Gamma=D_{n,k}(F)$ *is a geometric subspace of* $\Gamma'=A_{n,k}(F)$

Proof. Let l be a line of Γ'. We must show that if l has two of its incident points in Γ has all of its incident points in Γ. Suppose that p, q are the two incident points of l in Γ and r is a point incident with l. Then p and q form a TI (k+1)-space. But r is incident with l, then it a k-space contained in the TI (k+1)-space, so $r\in\Gamma$ and l is completely lying in Γ. □

4.3. THE MAIN THEOREM *Let* $\Gamma=(P, L)$ *be a geometry of type* $D_{n,k}(F)$, where $k \geq 2$, $n\geq k+3$ and F *is a field, then* Γ *can be embedded in a projective space and every regular geometric hyperplane arises from an embedding of* Γ *in a projective space.*

Proof. Let K be a regular geometric hyperplane of $D_{n,k}(F)$. Then by regularity there exist a geometric hyperplane H of $A_{n,k}(F)$ such that $K=H\cap D_{n,k}$. Since every hyperplane of $A_{n,k}$ arises from embedding. Then there exist an embedding μ' of $A_{n,k}$ into P(V) such that there exist H' a hyperplane of P(V) with $H=\mu'^{-1}(H')$. We know that $D_{n,k}$ is a subspace of $A_{n,k}$, then $\mu'(D_{n,k})$ is a subspace of P(V) say there exist a vector subspace W of V such that $\mu'(D_{n,k})=P(W)$, $H'\cap P(W)$ is a hyperplane of P(W). It follows that

$$K=\mu'^{-1}(H'\cap P(W)\cap\mu'(D_{n,k}))$$

$$=\mu'^{-1}(H'\cap P(W))\cap D_{n,k}$$

i.e., K arises from embedding. □

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