Hamilton – Jacobi Quantization of Christ – Lee Model

Prof. F. M. AL-Saba *
Dr. S. I. Muslih **
Mr. H. N. Harb***

الملخص

التكميم المساري للنظم المقيدة يتم مناقشتها باستخدام طريقة هاميلتون جاكوبي. نظام Christ – Lee يستم دراسته باستخدام هذه الطريقة حيث يتم إيجاد التكميم المساري لهذا النظام بدون استخدام أي شروط "gauge" خارجية.

Abstract

Path integral formulation of singular systems is investigated by the Hamilton–Jacobi method. The Christ–Lee model is studied by this method and the path integral quantization is obtained for this model without using any gauge fixing conditions.

* Department of Mathematics, Faculty of Education , Ain Shams University , Egypt.
** Department of Physics, AL – Azhar University , Gaza , Palestine.
*** Department of Mathematics, AL – Aqsa University , Gaza , Palestine.

1- Introduction

The first study of singular systems with constraints, were done by Dirac [1,2]. He showed that the Algebra of Poisson's brackets determines a division of constraints into two classes: The so called first class constraints and second class constraints. The first class constraints are those that have zero Poisson's brackets with all other constraints and second class constraints that have non-vanishing Poisson's brackets.

Recently, the Hamilton Jacobi method [3 – 6] was introduced to investigate singular systems. In this method the equations of motion are obtained as a total differential equations in many variables. If these equations are integrable then one can construct a cononical phase space coordinate without using any gauge fixing condition.

Quantization of constrained Hamiltonian systems can be achieved by means of operator quantization method or by path integral quantization. In this paper we consider the path integral quantization method of constrained systems.

This paper will be organized as follows: In section (2) we shall review the Hamilton – Jacobi method. In section (3) we treated the singular system by using the Hamilton – Jacobi method and quantized by using path integral quantization method.

2- The Hamilton –Jacobi method

The canonical method gives the set of Hamilton – Jacobi partial differential equation (H J P D E) as

$$H'_\alpha\left(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\alpha}\right) = 0 \quad \alpha, \beta = 0, n-r+1, \dots, n, \quad a = 1, \dots, n-r, \tag{1}$$

where

$$H'_\alpha = H_\alpha\left(t_\beta, q_a, p_a\right) + p_\alpha \quad , \tag{2}$$

and $H_\circ$ is defined as

$$H_\circ = p_a w_a + p_\nu \dot{q}_\nu \Big|_{p_\nu = -H_\nu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a) \quad . \tag{3}$$

$$\mu, \nu = n-r+1, \dots n$$

The equations of motion are obtained as a total differential equations in many variables as follows: [3,4]

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha , dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha , dq_\beta = \frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha \quad , \tag{4}$$

$$dz = -\left(H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \quad , \tag{5}$$

$$\alpha , \beta = 0, n-r+1, \ldots, n$$
$$a = 1, \ldots, n-r ,$$

where $z = S(t_\alpha , q_\alpha)$

The set of equation (4,5) is integrable if [5,6]

$$dH'_\circ = 0 \quad , \tag{6}$$

$$dH'_\mu = 0 \quad . \qquad \mu ; \nu = n-r+1, \ldots n \tag{7}$$

If conditions (6,7) are not satisfied identically, one considers them as new constraints and again tests the consistency condition. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates q_a and p_a as function of t_α, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians H'_α are considered as infinitesimal generators of canonical transformation given by parameters t_α respectively. In this case, the path integral representation may be written as: [6-12]

$$\langle out | s | in \rangle = \int \prod_{\alpha=1}^{n-p} dq^a dp^a \left[\exp i \left\{ \int_{t_\alpha}^{t'_\alpha} (-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha \right\} \right] \quad . \tag{8}$$

$$a = 1, \ldots, n-p$$
$$\alpha = 0, n-p+1, \ldots, n$$

One should notice that the path integral (8) is an integration over the canonical phase –space coordinates q_a and p_a.

3- The Hamilton – Jacobi treatment of Christ – Lee model

In this section we shall obtain the path integral quantization of Christ – Lee model. Let us consider the Lagrangian [5] as.

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+\lambda(x\dot{y}-y\dot{x})+\tfrac{1}{2}\lambda^2(x^2+y^2)-V(x^2+y^2)\ . \tag{9}$$

The moment corresponding to the generalized coordinates may be written as.

$$p_x=\frac{\partial L}{\partial \dot{x}}=\dot{x}+\lambda y\ , \tag{10}$$

$$p_y=\frac{\partial L}{\partial \dot{y}}=\dot{y}+\lambda x\ , \tag{11}$$

$$p_\lambda=0\ . \tag{12}$$

From equations (2) and (3) we can solve $\dot{x}$ and $\dot{y}$ in terms of p_x, p_y respectively

$$\dot{x}=p_x-\lambda y=w_1\ , \tag{13}$$

$$\dot{y}=p_y-\lambda x=w_2\ . \tag{14}$$

The Canonical Hamiltonian is defined as

$$H_\circ=p_x w_1+p_y w_2-L\ . \tag{15}$$

Making use (13) and (14) one gets

$$H_\circ=\tfrac{1}{2}\left(p_x^2+p_y^2\right)+\lambda\left(p_y x-p_x y\right)+V\left(x^2+y^2\right)\ . \tag{16}$$

Now, the canonical formulation gives us the set of Hamilton–Jacobi partial defferential equation (HJPDE) as:

$$H_{\circ} = p_{\circ} + H_{\circ} = 0 \ , \tag{17}$$

$$H' = p_{\lambda} = 0 \ , \tag{18}$$

where

$$H_{\circ} = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + \lambda\left(p_y x - p_x y\right) + V\left(x^2 + y^2\right) . \tag{19}$$

Now, the equations of motion are defined as.

$$dx = \frac{\partial H_{\circ}}{\partial p_x} dt + \frac{\partial H'}{\partial p_x} d\lambda = \left(p_x - \lambda_y\right) dt \ , \tag{20}$$

$$dy = \frac{\partial H'_{\circ}}{\partial p_y} dt + \frac{\partial H'}{\partial p_y} d\lambda = \left(p_y - \lambda_y\right) dt \ , \tag{21}$$

$$dp_x = -\frac{\partial H'_{\circ}}{\partial x} dt - \frac{\partial H'}{\partial x} d\lambda = -\left(\lambda p_y + \frac{\partial V}{\partial x}\right) dt \ , \tag{22}$$

$$dp_y = -\frac{\partial H'_{\circ}}{\partial y} dt - \frac{\partial H'}{\partial y} d\lambda = -\left(-\lambda p_x + \frac{\partial V}{\partial y}\right) dt \ , \tag{23}$$

$$dp_{\lambda} = -\frac{\partial H'_{\circ}}{\partial \lambda} dt - \frac{\partial H'}{\partial \lambda} d\lambda = -\left(p_y x + p_x y\right) dt \ , \tag{24}$$

$$dp_{\circ} = -\frac{\partial H'_{\circ}}{\partial t} dt + \frac{\partial H'}{\partial t} d\lambda = 0 \ . \tag{25}$$

To check whether the set of equation (20-25) is integrable or not, let us consider the variations of (17,18). In fact

$$dH'_{\circ} = dp_{\circ} + dH_{\circ} = \left(xp_y - yp_x\right) d\lambda \ . \tag{26}$$

Since $dH'_{\circ}$ doesn't vanish identically then, one considers it as a new constraint.

$$F_1 = xp_y - yp_x \ . \tag{27}$$

Since F_1 is not identically zero, we consider it as a new constraint and its variation is identically zero.

Making use of (5) and (8) we obtain the canonical action integral as:

$$Z = \int\left[\tfrac{1}{2}\left(p_x^2 + p_y^2\right) - V\left(x^2 + y^2\right)\right] dt \ , \tag{28}$$

Then, the path integral quantization is given by

$$\langle out|s|in\rangle = \int \prod dx dy dp_x dp_y \exp\left[i \int_{t}^{t'} \left[\tfrac{1}{2}\left(p_x^2 + p_y^2\right) - V\left(x^2 + y^2\right)\right] dt\right]. \tag{29}$$

Now to obtain the path integral quantization of the Christ – Lee model in polar Coordinates. Let us consider the following canonical transformations [13]

$$x = \rho \cos\phi \ \ , y = \rho \sin\phi \ , \tag{30}$$

$$p_x = \Pi_\rho \cos\phi - \frac{\Pi_\phi}{\rho}\sin\phi \ \ , \ p_y = \Pi_\rho \sin\phi + \frac{\Pi_\phi}{\rho}\cos\phi \ , \tag{31}$$

and taking into account the constraints (13,14), we obtain the action integral as:

$$Z = \int dt\left[\Pi_\rho \dot{\rho} - \tfrac{1}{2}\Pi_\rho^2 + V\left(\rho^2\right) + \tfrac{1}{8\rho^2}\right]. \tag{32}$$

The path integral representation of the model (9) is then given by

$$< \rho, \rho', t, t' >= \int d\rho \, d\Pi_\rho \ \exp \int dt\left[\Pi_\rho \dot{\rho} - \tfrac{1}{2}\Pi_\rho^2 + \tfrac{1}{8\rho^2}\right] . \tag{33}$$

This result is in exact agreement with those Previously obtained by Muslih [14] and Rabei [15].

4- Conclusion

In this work, we have investigated the path integral quantization of constrained system using the Hamilton – Jacobi method. In this approach, the equations of motion are obtained as total differential equations in many variables. If the system is integrable, then we can obtain the canonical physical variables and the path integral as an integration over these physical variables without using any gauge fixing conditions.

In the Christ – Lee mode (9), this system is integrable and the path integral is obtained as an integration over the canonical phase spase coordinates (x, p_x) and (y, p_y). Besides when we use suitable canonical transformations [13] and taking into account the integrability conditions, we obtain the canonical physical variable (ρ, Π_ρ) and the canonical path integral equation as an integration over (ρ, Π_ρ) without using any gauge fixing conditions.

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