

Embedding and hyperplane of point-line geometry of type $D_{n,k}(F)$

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الملخص

أثبت Shult [7] لفضاء Grassmann المنتهي القابل للطمر أن كل مستوى فوقي يأتي من الطمر. في هذا البحث نبرهن أن هندسة نقطة-خط من نوع $D_{n,k}$ حيث $k \geq 2$ ، قابلة للطمر في فضاء إسقاطي وأن كل مستوى فوقي عادي يأتي من ذلك الطمر. ، $n \geq k+3$

Abstract

Shult [7] proved that every hyperplane of embeddable Grassmann space of finite rank, arises from an embedding. In this paper it is proved that the point-line geometry of type $D_{n,k}$ ($k \geq 2, n \geq k+3$) can be embedded in projective space and every regular hyperplane arises from an embedding using the result in [7].

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Introduction

Many authors are interested in embeddability of some geometries in a projective space. Lynn Batten [4] found an intimate relationship between affine and projective planes by showing that any affine plane is embeddable in projective plane. Cooperstein and Shult [3] showed that any geometric hyperplane arises from an embedding for the following Lie-incidence geometries: $A_{n,2}$, $D_{5,5}$ and $E_{6,1}$.

A great effort has been spent by Shult [8] showing that the half-spin geometry of type $D_{n,n}$ is embeddable in a projective space. Ronan Theory [5] presented a basic tool for certain strong parapolar spaces to show that whether the hyperplanes arise from an embedding. It has been proved in [9] the embeddability of certain class of geometries $D_{n,2}$, $D_{n,2}$ and $D_{n,3}$. In this paper we use the result in [7] to prove the embeddability of the point-line geometry $D_{n,k}$, $k \geq 2$ and $n \geq k+3$.

A *subspace* of a point-line geometry $\Gamma=(P, L)$ is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X . $\langle X \rangle$ means the intersection over all subspaces containing X , where $X \subseteq P$. Lines incident with more than two points are called *thick* lines, those incident with exactly two points are called *thin* lines.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there exist a chain of distinct subspaces $\emptyset \neq X_0 \subset X_1 \subset \dots \subset X_n$ such that X_i is singular for each i , $X_i \neq X_j$, $i \neq j$. For example $\text{rank}(\emptyset)=-1$, $\text{rank}(\{p\})=0$ where p is a point and $\text{rank}(l)=1$ where l a line.

In a point-line geometry $\Gamma=(P, L)$, a path of length n is a sequence of $n+1$ (x_0, x_1, \dots, x_n) where, (x_i, x_{i+1}) are collinear, x_0 is called the initial point

and x_n is called the end point. A *geodesic* from a point x to a point y is a path of minimal possible length with initial point x and end point y . We denote this length by $d_\Gamma(x, y)$.

A geometry Γ is called *connected* if and only if for any two of its points there is a path connecting them. A subset X of P is said to be *convex* if X contains all points of all geodesics connecting two points of X .

A *polar space* is a point-line geometry $\Gamma=(P, L)$ satisfying the axiom: For each point-line pair (p, l) with p not incident with l ; p is collinear with one or all points of l , that is $|p^\perp \cap l|=1$ or else $p^\perp \supset l$. Clearly this axiom is equivalent to saying that p^\perp is a geometric hyperplane of Γ for every point $p \in P$.

A point-line geometry $\Gamma=(P, L)$ is called a *projective plane* if and only if it satisfies the following conditions [5]:

- (i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line,
- (ii) every two lines intersect in one point,
- (iii) there are four points no three of them are on a line.

A point-line geometry $\Gamma=(P, L)$ is called a *projective space* if the following conditions are satisfied:

- (i) every two points lie exactly on one line ,
- (ii) if l_1, l_2 are two lines $l_1 \cap l_2 \neq \emptyset$, then $\langle l_1, l_2 \rangle$ is a projective plane. ($\langle l_1, l_2 \rangle$ means the smallest subspace of Γ containing l_1 and l_2 .)

A point-line geometry $\Gamma=(P, L)$ is called a *parapolar space* if and only if it satisfies the following properties:

- i- Γ is a connected gamma space,
- ii- For every line l ; l^\perp is not a singular subspace,

iii- For every pair of non-collinear points x, y ; $x^\perp \cap y^\perp$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If x, y are distinct points in P , and if $|x^\perp \cap y^\perp|=1$, then (x, y) is called a *special pair*, and if $x^\perp \cap y^\perp$ is a polar space, then (x, y) is called a *polar pair* (or a *symplectic pair*). A parapolar space is called a *strong parapolar space* if it has no special pairs.

Embeddings.[6] Let $\Gamma=(P, L)$ be a point-line geometry, that is, an incidence system of points P and lines L , such that each line is viewed as a set of points. A *projective embedding* of a point-line geometry Γ into the projective space $\mathbf{P}(V)$ of all proper subspaces of the vector space V is an injective mapping $e: P \rightarrow$ projective points of $\mathbf{P}(V)=1$ -spaces of V such that

- i- $e(l)$ is a projective line for each l of L , and
- ii- the image points $e(P)$ span $\mathbf{P}(V)$.

Such an embedding is denoted by the symbol $e: \Gamma \rightarrow \mathbf{P}(V)$.

A geometric hyperplane of a point-line geometry is a proper subspace which meets each line non-trivially. Let $e: \Gamma \rightarrow \mathbf{P}(V)$ be a projective embedding of the point line geometry $\Gamma=(P,L)$. Suppose \mathbf{H} is a projective hyperplane of $\mathbf{P}(V)$, the set $H(\mathbf{H}):=\{x \in P \mid e(x) \in \mathbf{H}\}$ is a hyperplane of Γ . A hyperplane H of Γ is said to arise from the embedding e if and only if it has the form $H=H(\mathbf{H})$.

Morphisms of embeddings. Let $\tau: V \rightarrow W$ be a semilinear transformation of vector spaces. This induces a partial mapping of the corresponding projective spaces $\mathbf{P}(V)$ and $\mathbf{P}(W)$, sending points of $\mathbf{P}(V)$ not contained in $\ker \tau$ in $\mathbf{P}(V)$ to projective points of $\mathbf{P}(W)$. If $e: \Gamma \rightarrow \mathbf{P}(V)$ is a projective embedding of the point-line geometry Γ , then composition with the partial

map τ can yield a new embedding $e\tau$ if and only if it satisfies the following conditions:

i- τ is a surjective semilinear transformation, and

ii- for any point p and q of Γ , $\ker \tau$ meets the subspace $\langle e(p), e(q) \rangle$ at the zero subspace of V . We call the transfer from embedding e to embedding $e\tau$ *a morphism of embeddings* and write $e \rightarrow e\tau$. In general if we insist that $e \rightarrow e'$ is a morphism of embedding, it means that $e' = e\tau$ for an appropriate semilinear transformation τ . If $\ker \tau$ is the zero vector space, e and e' are said to be *equivalent embeddings*.

An embedding $u: \Gamma \rightarrow \mathbf{P}(V)$ is said to be *relatively universal* if and only if the existence of a morphism $w \rightarrow u$ implies w is equivalent to u .

2- Old results and Notations

Shult has proved in the following theorem that every hyperplane of embeddable Grassmannspace of finite rank, arises from an embedding. For characterization and construction of $A_{n,k}$ see [1-2].

2.1 Theorem [7]. Let $\Gamma = (P, L)$ be a Grassmann space of type $A_{n,k}(F)$, where F is a field, $1 < k < n$. Then every geometric hyperplane of Γ arises from the universal embedding of Γ in $\mathbf{P}(W)$ where W is the k -fold wedge product of the n -space V over F with itself.

For embeddings of different Lie incidence geometries, there are three methods showing that every hyperplane arises from embedding that have been successful.

Method 1. (Circuitry) A circuit $C = (x_0, x_1, \dots, x_{n-1})$ in the collinearity graph $X = (P-H, \sim)$ is said to be *decomposable* if it is the sum of circuits C_0, C_1, \dots , in X such that each C_i lies in some symplecton S_i of the parapolar space Γ .

We say that C is *minimal indecomposable* if C is not decomposable and all circuits of smaller length are decomposable.

Using results of Ronan [5], for a strong parapolar space, it is sufficient to show that for any hyperplane H of Γ , that in the subgraph of (P, \sim) induced on $P-H$, every circuit C is a sum of triangles and 4-circuits.

Method 2. (Inductive construction of a functional) If $e: \Gamma \rightarrow P(V)$ is the embedding, we wish to show that for each hyperplane H of Γ , there exists a functional $h: V \rightarrow F$ of V such that $e \cdot h$ vanishes on H but never vanishes on $P-H$. Inductively there exists a family S of subgeometries of Γ belonging to a parameterized family of geometries containing Γ and such that the restriction of e to S in S is still relatively universal. Then for each $S \in S$, there exists by induction a functional

$$H_S : \langle e(S) \rangle := W_S \rightarrow F$$

Which vanishes on $e(S-H)$ (if the latter is empty, of course, $h_S=0$). This method was used in [7].

Method 3. (The direct sum method) This method also uses induction indifferent way. We assume as before that $e: \Gamma \rightarrow P(V)$ is relatively universal. Again Γ is assumed to belong a family of geometries parameterized by an integer-valued function τ . The theorem being proved asserts that for some function $\delta: Z \rightarrow Z$, if $\dim V \geq \delta(\tau(\Gamma))$, then $\dim V = \delta(\tau(\Gamma))$ and every hyperplane arises from embedding. The geometries Γ must have the property that they always possess two subgeometries S_1 and S_2 in the same parameterized family satisfying

i- $\langle S_1, S_2 \rangle_\Gamma = P$, and

ii- $\delta(\tau(S_1)) + \delta(\tau(S_2)) \leq \delta(\tau(\Gamma))$.

Induction on the S_i immediately yields the fact that $V = \langle e(S_1) \rangle \oplus \langle e(S_2) \rangle$ and that $\dim \langle e(S_i) \rangle_V = \delta(\tau(S_i))$. By induction, the hyperplane H can be assumed to meet each S_i properly at a hyperplane H_i of S_i . One sets X to be all points x of H such that $e(x)$ is not in the subspace $U := \langle e(H_1) \rangle \oplus \langle e(S_2) \rangle$ of codimension 2 in V . For $x, y \in X$, one write

$$x \sim y \text{ if and only if } U \oplus \langle x \rangle = U \oplus \langle y \rangle.$$

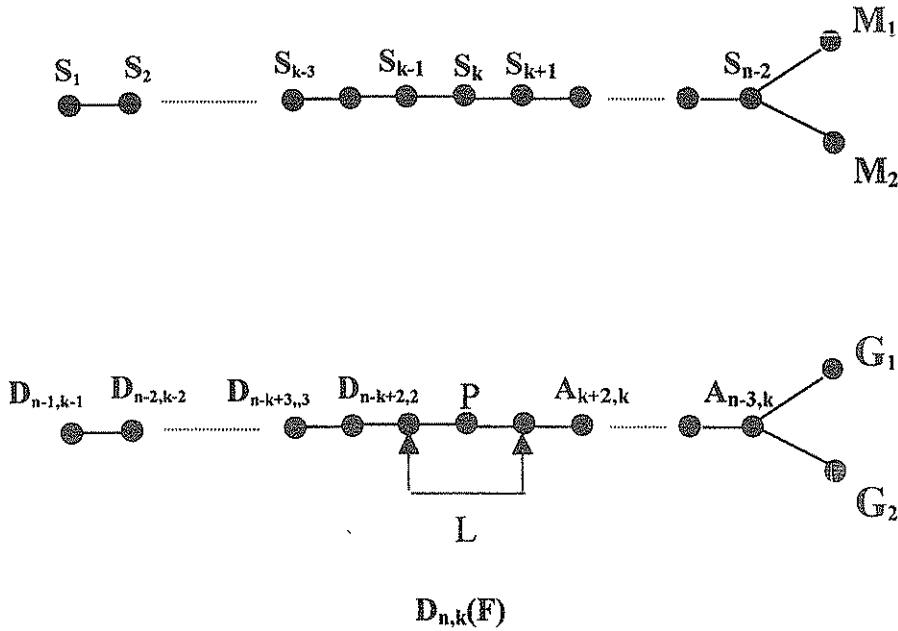
The rest of the proof consists in showing that the graph (X, \sim) is connected.

Summarizing known results, we have:

Geometry	Reference	Method
$E_{6,1}, D_{5,5}$ and $A_{n,2}$	Cooperstein and Shult [3]	Circuitry
Half-spin geometry, $D_{n,n}$	Shult [8]	Direct sum
$A_{n,k}, 2 \leq [(n+1)/2]$	Shult [7]	Functional method

We present some facts about the point-line geometry of type $D_{n,k}$ that can be found in [9, 10].

Construction of $D_{n,k}(\mathbb{F})$ [10]. Consider the polar space $\Delta = \Omega^+(2n, \mathbb{F})$ that comes from a vector space V of dimension $2n$ over a finite field $\mathbb{F} = \text{GF}(q)$ with a symmetric hyperbolic bilinear form B . S_i is the set of all totally isotropic i -dimensional subspaces of V ; $1 \leq i \leq n-2$. The two classes M_1, M_2 consist of maximal totally isotropic n -dimensional subspaces. Two n -spaces fall in the same class if their intersection is of odd dimension.



The geometry of type $D_{n,k}(F)$ is the point-line geometry (P, L) , whose set of points P is the collection of all k -dimensional subspaces of the vector space V , and whose lines are the pairs (A, B) where A is $(k-1)$ -dimensional subspace of $(k+1)$ -subspace B —that is, the set of $(k-1, k+1)$ -subspace of V . A point C is incident with a line (A, B) if and only if $A \subset C \subset B$ as a subspaces of V .

To define the collinearity, let C_1 and C_2 be two points (the points are the T.I k -spaces), then C_1 is collinear to C_2 if and only if the intersection of $C_1 \cap C_2 = (k-1)$ -space and $\langle C_1, C_2 \rangle = (k+1)$ -space.

The elements of the classes G_1 and G_2 are Grassmannian geometries of type $A_{n-1,k}$.

There are two kinds of symplecta. The first kind is the convex polar spaces $A_{3,2}$ that represents the $(k-2, k+2)$ subspaces of V . Then symplecton S of kind $A_{3,2}$ is the set of T.I k -dimensional spaces that contain the T.I $(k-2)$ -

dimensional space and contained in the TI $(k+2)$ -dimensional space. (2)
 The second kind of symplecta is the convex polar spaces of type $D_{n-k+1,1}$ that represents the collection of all TI $(k-1)$ -subspaces of V . Thus this kind of symplecta is defined as the collection of all TI k -subspaces of V that contain such TI $(k-1)$ -spaces.

2.2 Proposition [9]. *Let $\Gamma=(P, L)$ be the geometry of type $D_{n,k}(F)$. Thus:*

- i- Γ is of diameter $k+1$,
- ii- Γ is a weak parapolar geometry.

3- The main result

Before proving the main theorem we need to present the definition of the regular hyperplane and we show that the point line geometry $D_{n,k}$ is a geometric subspace of the Grassmann geometry $A_{n,k}$.

3.1 Definition [9]. Let Γ be a sub-geometry of a geometry Γ^λ . Let H be a hyperplane of Γ . If there exist a hyperplane H^λ of Γ^λ such that $H=H^\lambda \cap \Gamma$ we say that H is a regular hyperplane of Γ with respect to Γ^λ . Otherwise we say that H is irregular with respect to Γ^λ .

For example if H^λ is a hyperplane of Γ^λ , then $K=H^\lambda \cap \Gamma$ is a hyperplane of Γ implies that K is regular with respect to Γ^λ .

3.2. Lemma $\Gamma=D_{n,k}(F)$ is a geometric subspace of $\Gamma^\lambda=A_{n,k}(F)$

Proof. Let l be a line of Γ^λ . We must show that if l has two of its incident points in Γ has all of its incident points in Γ . Suppose that p, q are the two incident points of l in Γ and r is a point incident with l . Then p and q form a TI $(k+1)$ -space. But r is incident with l , then it a k -space contained in the TI $(k+1)$ -space, so $r \in \Gamma$ and l is completely lying in Γ .

4.3. THE MAIN THEOREM *Let $\Gamma=(P, L)$ be a geometry of type $D_{n,k}(F)$, where $k \geq 2$, $n \geq k+3$ and F is a field, then Γ can be embedded in a projective space and every regular geometric hyperplane arises from an embedding of Γ in a projective space.*

Proof. Let K be a regular geometric hyperplane of $D_{n,k}(F)$. Then by regularity there exists a geometric hyperplane H of $A_{n,k}(F)$ such that $K=H \cap D_{n,k}$. Since every hyperplane of $A_{n,k}$ arises from embedding. Then there exists an embedding μ^{-1} of $A_{n,k}$ into $P(V)$ such that there exists H^{\setminus} a hyperplane of $P(V)$ with $H=\mu^{-1}(H^{\setminus})$. We know that $D_{n,k}$ is a subspace of $A_{n,k}$, then $\mu^{-1}(D_{n,k})$ is a subspace of $P(V)$ say there exists a vector subspace W of V such that $\mu^{-1}(D_{n,k})=P(W)$, $H^{\setminus} \cap P(W)$ is a hyperplane of $P(W)$. It follows that

$$\begin{aligned} K &= \mu^{-1}(H^{\setminus} \cap P(W) \cap \mu^{-1}(D_{n,k})) \\ &= \mu^{-1}(H^{\setminus} \cap P(W)) \cap D_{n,k} \end{aligned}$$

i.e., K arises from embedding.

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