GATEDNESS PROPERTY in a CLASS of LIE INCIDENCE GEOMETRIES*

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ملخص

في هذا البحث أثبتنا أن خاصية الشكل الخماسي¹ الضعيف تكافئ خاصية التبويب² في الفراغات البارقطية الشبه قوية³ .

١. خاصية الشكل الخماسي الضعيف إذا كانت ن٠ ، ن١،ن٢،ن٣،ن٤ شكل خماسي ولا يكون هناك خطوط بين رؤوسه ، وكانت نل⊥ ∩ نل+٢⊥ ∩ نل+٣⊥ ≠ Φ لكل ل = ٠،١،٢،٣،٤ (بواقي ٥) .

٢. إذا كانت أ نقطة ، س سمبلكته وكان أ⊥ ∩ س = {ب} حيث ب نقطة فان المعادلة تتحقق م(أ،ج) = م(أ،ب) + م(ب،ج) حيث ج أي نقطة اختيارية داخل السمبلكته – م(أ،ج) هي عبارة عن المسافة بين أجـ .

٣. يسمي الفراغ البارقطي إذا كان لكل شكل خماسي (س٠، س١، س٢، س٣، س٤) ليس لـه أقطار < سل،سل+١ سمبلكته > أي أن يكون الشكل الخماسي ليس له زوج خاص .

كما طبقنا هذه النتائج علي بعض الهندسات مثل $D_{5,5}(F)$, $E_{6,4}(F)$, $F_{4,1}(F)$, $E_{6,1}(F)$, $E_{7,7}(F)$, and $E_{8,1}(F)$ حيث F حقل .

ABSTRACT

We show that the weak pentagon property (wp)$^{(1)}$ is equivalent to the gatedness property (G)$^{(2)}$ in the class of semi-strong parapolar spaces$^{(3)}$.

$^{(1)}$(wp) **(Weak Pentagon Property).** *In a parapolar space* Γ, *we say that the weak pentagon property holds in* Γ, *if in each pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *with no diagonals*; $x_i^{\perp} \cap x_{i-2}^{\perp} \cap x_{i-3}^{\perp} \neq \varnothing$, *for all* $i = 0, 1, 2, 3, 4$. (*indices are taken mod* 5).

$^{(2)}$(G) **(Gatedness Property).** *If* (x, S) *is a point-symplecton pair such that if* $|x^{\perp} \cap S| = 1$, *then* (x, S) *is gated at the unique point of* $(x^{\perp} \cap S) = \{g\}$. *That is for every point* $y \in S$, *we have*

$$d_\Gamma(x, y) = d_\Gamma(x, g) + d_s(g, y)$$

$^{(3)}$*A parapolar space* Γ *is called semi-strong parapolar space if in each pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *with no diagonals* $\langle x_i, x_{i-2}\rangle$ *is a symplecton* (*indices are taken mod* 5) *for all* $i = 0, 1, 2, 3, 4$. *In other words pentagons have no special pair.*

We also apply these results to certain families of spaces. In particular, we apply the above result to the class of Lie incidence geometries: $D_{5,5}(F)$, $E_{6,4}(F)$, $F_{4,1}(F)$, $E_{6,1}(F)$, $E_{7,7}(F)$, and $E_{8,1}(F)$ for some field F.

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1 Basic Definitions

A point-line geometry Γ = (P, L) is a pair of sets; P is called the set of points and L is called the sets of lines, where the members of L are just subsets of P. If p is a point that belongs to a line l, we say that p lies on l or l passes through p or p is incident with l.

In point-line geometry Γ = (P, L), we say that two points $p, q \in$ P are *collinear* if they are incident with a common line, and we write $p \sim q$.

For any point p in point-line geometry Γ = (P, L), define

$$p^{\perp} = \{p\} \cup \{q \in P \mid p \sim q\}.$$

For any set of points $X \subset P$, define $X^{\perp} = \cap\{p^{\perp} \mid p \in X\}$.

$Rad(X) = X \cap X^{\perp}$. In particular,
$Rad(\Gamma) = P^{\perp} = \{q \in P \mid p \text{ is collinear to } q \text{ for all } p \in P\}$.

Γ = (P, L) is called *a linear (or singular)* space if each pair of distinct points lies exactly on one line. Γ is called *a partial linear* if each pair of points lies on at most one line.

A subspace of a point-line geometry Γ = (P, L) is a subset X of points such that if a line l has at least two points of X then l lies entirely in X.

A path of length k from x_0 to x_k is a sequence of $k + 1$ points $x_0, x_1, \ldots, x_k$ such that x_{i-1} is collinear to x_i, $i = 1, 2, 3, \ldots\ldots\ldots\ldots, k\text{-}1$.

A geodesic is a shortest path between two points. We define the distance function

$$d_\Gamma : P \times P \to Z, \text{ by}$$
$$d_\Gamma(x, y) = \text{the length of any geodesic from } x \text{ to } y.$$

A subspace X is called convex if it contains all geodesics between any two points of X.

The smallest subspace containing a set X is called the *subspace generated by* X and denoted by $\langle X \rangle$.

A subspace X is called connected if for each pair of points there is a path that connects them and lies entirely in X.

The singular rank of a space is the maximal number n for which there exists a chain of distinct subspaces

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_k$$

such that X_i is singular for each i, with $X_i \neq X_j$, $i \neq j$. In this sense, the empty set has rank equals -1, rank(p) = 0, for any point p, rank(l) = 1, for any line l.

Let p be a point in P, we define $\Delta_k(p)$ and $\Delta_k^*(p)$ as follows:

$\Delta_k^*(p) = \{x \in P \mid x \text{ is of distance at most } k \text{ from } p\}$,

$\Delta_k(p) = \{x \in P \mid x \text{ is of distance exactly } k \text{ from } p\}$.

A *geometric hyperplane* is a subspace that meets every line of the space.

2 Some Families of Spaces

A point-line geometry is called *projective plane* if it satisfies the following conditions:

(i) Γ is a linear space, i.e., every two distinct points $x, y \in P$ lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points, no three of which are on one line.

A point-line geometry $\Gamma = (P, L)$ is called *projective space* if the following conditions are satisfied

(i) every two points lie exactly on one line.

(ii) if l_1, l_2, are two intersecting lines then $\langle l_1, l_2 \rangle$ is a projective plane.

A *gamma space* is a point -line geometry such that for every point-line pair (p, l) with p not on l, p is collinear to no points of l, one point of l or all points of l.

It is easy to show that a gamma space is a space in which $p^{\perp}$ is a subspace, for every point $p \in P$.

In 1959, Veldkamp has published a paper [VE], in which he axiomatically characterized certain families of geometries (Polar spaces) related to polarities in a vector spaces. In 1974, Buekenhout and Shult, in [BS], proved that the axiomatization can be dramatically simplified. They used the one-or-all axiom.

A polar space is a point-line geometry $\Gamma = (P, L)$ that satisfies the following Buekenhout-Shult one-or-all axiom

for each point p not incident with a line l, p is collinear with one point of l or all points of l.

It is easy to see that a polar space is a space in which $p^{\perp}$ is a geometric hyperplane, for every point $p \in P$.
A polar space $\Gamma = (P, L)$ is called *degenerate* if $rad(\Gamma) \neq \varnothing$, otherwise it is called *non-degenerate*.

3 Classical Examples of Finite Polar Spaces

Let V be a vector space over a finite field $F = GF(q)$, q is a prime power.

1. *Symplectic Geometry* $W_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a symplectic bilinear form B. In this case n is even and the polar space is of rank $n/2$.

2. *Hyperbolic Geometry* $\Omega^{+}_{n}(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a hyperbolic bilinear form B. In this case n is even and the polar space is of rank $n/2$.

3. *Elliptic Geometry* $\Omega^{-}_{n}(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a elliptic bilinear form B. In this case n is even, the polar space is of rank $(n/2) - 1$.

4. *Orthogonal Geometry* $\Omega_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for an orthogonal bilinear form B. In this case n is odd, the polar space is of rank $(n-1)/2$.

5. *Hermitian Geometry* $H^{*}_{n}(q^2)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which

B(x, x) = 0, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which B(x, y) = 0, for a Hermitian bilinear form B. In this case n is even, the polar space is of rank $n/2$.

6. *Hermitian Geometry* $H_n(q^2)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which B(x, x) = 0, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which B(x, y) = 0, for a Hermitian bilinear form B. In this case n is odd, the polar space is of rank $(n-1)/2$.

Polar spaces of rank at least 3 have been classified [BS], [B], [JP], [BSP]. It turned out that those spaces in which each line contains at least 3 points and rank at least 4 are classical polar spaces.

A point-line geometry is called a *parapolar space* if it satisfies the following conditions.

(i) Γ is a connected gamma space with no lines of cardinality two,

(ii) for every line; $l^{\perp}$ is not a singular space,

(iii) for every pair of points x, y of distance 2, $x^{\perp}\cap y^{\perp}$ is either a point, or non-degenerate polar space of rank at least 2.

A parapolar space is called *strong parapolar space* if for every pair of points x, y of distance 2, $x^{\perp}\cap y^{\perp}$ is non-degenerate polar space of rank at least 2.

COOPERSTEIN'S THEORY. In [Coo], Cooperstein has shown that if (P, L) is a parapolar space then for every pair of points p, q of distance 2, the convex closure $\langle p, q\rangle$ is a non-degenerate convex polar space whose rank is one more than the rank of $p^{\perp}\cap q^{\perp}$. Moreover he has shown that all maximal singular subspaces of $\langle p, q\rangle$ are projective spaces. Such a convex polar space $\langle p, q\rangle$ is called a symplecton. Convexity of such polar spaces forces the following very important property:

for any two distinct symplecta S_1, S_2 and a point p; $p^{\perp}\cap S_1$ and $S_1\cap S_2$ are a projective subspaces of S_1(or S_2).

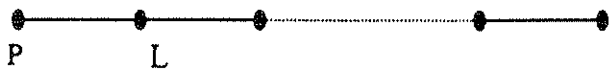
4 Lie Incidence Geometries

A point-line geometry $\Gamma = (P, L)$ whose points P are the cosets of a maximal parabolic subgroup of Lie type, and whose lines L are the cosets of the parabolic subgroup P' corresponding to the collection of all nodes in the dynkin diagram adjacent to the unique node corresponding to P.

Next we will give some classical examples of finite spaces

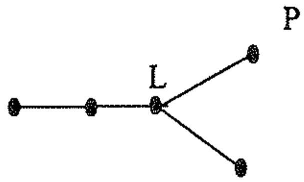
1. *Projective Spaces*

$A_{n,1}(F)$:



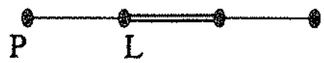
2. *Half-spin Geometry*

$D_{5,5}(F)$:



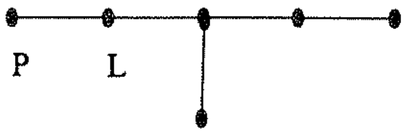
3. *Metasymplectic spaces*

$F_{4,1}(F)$:

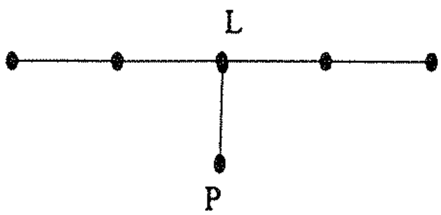


4. *Exceptional Geometries*

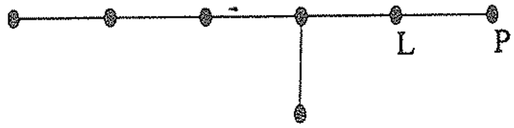
$E_{6,1}(F)$:



$E_{6,4}(F)$:



$E_{7,7}(F)$:



$E_{8,1}(F)$:

5 Properties of Parapolar Spaces

In a parapolar space (P, L), if x_0, x_1, x_2, x_3, x_4, are five points in P, we say that $(x_0, x_1, x_2, x_3, x_4)$ is **a pentagon** if x_i is collinear with x_{i+1}. We say that $(x_0, x_1, x_2, x_3, x_4)$ is **a pentagon with no diagonals**, if $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon such that x_i is neither collinear with x_{i+2} nor x_{i+3} $i = 0, 1, 2, 3, 4$ (indices are taken mod 5).

(wp) (**Weak Pentagon Property**). *In a parapolar space* Γ, *we say that the weak pentagon property holds in* Γ, *if in each pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *with no diagonals;* $x_i^{\perp} \cap x_{i+2}^{\perp} \cap x_{i+3}^{\perp} \neq \emptyset$ *for all* $i = 0, 1, 2, 3, 4$. (*indices are taken mod* 5).

(P) (**Pentagon Property**). *If* $(x_0, x_1, x_2, x_3, x_4)$ *is a pentagon with no diagonals, then* x_i *is* collinear *to one point on the opposite line* $x_{i+2}x_{i+3}$ *with* $i = 0, 1, 2, 3, 4$ (*indices are taken mod* 5).

Definition. *A parapolar space* Γ *is called semi-strong parapolar space if in each pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *with no diagonals* $\langle x_i, x_{i+2} \rangle$ *is a symplecton (indices are taken mod* 5) *for all* $i = 0, 1, 2, 3, 4$. *In other words pentagons have no special points.*

This class was first introduced in [At2]. In a parapolar space (wp) implies that the space is semi-strong parapolar space. Strong parapolar space is semi-strong parapolar space. So semi-strong parapolar space is a class of parapolar spaces in "between" parapolar spaces and strong parapolar spaces.

(**Gated**) *A subset S in a point-line geometry* $\Gamma = (P, L)$ *is said to be gated with respect to a point x not in S, if there exist a point g (the gate with respect to x) such that for all* $y \in S$, *we have*

$$d_\Gamma(x, y) = d_\Gamma(x, g) + d_S(g, y)$$

(G) (**Gatedness Property**). *In a point-line geometry* $\Gamma = (P, L)$, *If* (x, S) *is a point-symplecton pair such that if* $|x^\perp \cap S| = 1$ *then* (x, S) *is gated at the unique point* $(x^\perp \cap S) = \{g\}$. *That is for every point* $y \in S$, *we have*:

$$d_\Gamma(x, y) = d_\Gamma(x, g) + d_S(g, y)$$

In a parapolar space $\Gamma = (P, L)$ if $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon with no diagonals, and if x_0 is collinear to a point on the opposite side (say z) then we have $|x_r^\perp \cap x_r^\perp| \geq 2$. By Cooperstein Theory, $\langle x_0, x_r \rangle$ is a symplecton (say S). Now S contains the two points z, x_3, and since S is a subspace it must contain the whole line x_2x_3, therefore $x_2 \in S$. Since S is a convex subspace then $x_1 \in S$. Hence the whole pentagon $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton.

On the other hand if a pentagon with no diagonals $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton S then because the space S is a polar space; x_0 is collinear to one point or all points of line x_2x_3. However x_0 can not be collinear to all points of x_2x_3 since the pentagon has no diagonals. Thus x_0 is collinear to one point on the other side. This argument will give us the following:

In a parapolar space, (p) is equivalent to saying that pentagons with no diagonals lie in a symplecton.

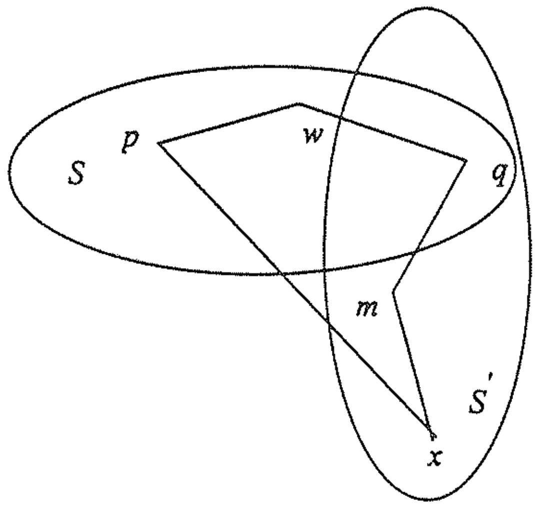
6 Present Results

EL-Atrash has shown, in [At1], that in the class of strong parapolar space the property of having the intersection of two symplecta is never a point for all distinct pairs of symplecta is a sufficient condition for the gatedness property (G). Here we prove the same result in the wider class of semi-strong parapolar space. In proving this result, we follow the same way which is used in [At1].

Proposition 6.1. *Let* $\Gamma = (P, L)$ *be a semi-strong parapolar space. Suppose that rank* $(S_1 \cap S_2) \neq 0$ *for any two symplecta* S_1, S_2 *then* (G) *holds in* Γ.

Proof: Let (x, S) be a non-incident point-symplecton pair such that $x^\perp \cap S = \{p\}$, a single point. We want to prove that $d_\Gamma(x, y) = 1 + d_S(p, y)$ for every $y \in S$. Since S is a polar space, we have two cases, either

$y \sim p$ i.e., $d_S(p, y) = 1$ or $d_S(p, y) = 2$. For the first case we want to *show* that $d_\Gamma(x, y) = 2$. Since $x^\perp \cap S$ is a unique point $\{p\}$ by hypotheses, therefore $d_\Gamma(x, y) \neq 1$, hence $d_\Gamma(x, y) = 2$. Now for the second case. Let q be a point in S such that $d_S(p, q) = 2$. We need to show $d_\Gamma(x, q) = 3$. Since $x^\perp \cap S = \{p\}$ is a single point, x is not collinear with q therefore we may, by way of contradiction, assume that $d_\Gamma(x, q) = 2$. Since $d_S(p, q) = 2$; p is not collinear with q. And x is not collinear with q by hypotheses. Choose $m \in x^\perp \cap q^\perp$. $m \neq p$, since q is not collinear to p, also $m \notin S$, since otherwise $x^\perp \cap S$ would be more than one point. In fact $m \notin p^\perp$ by convexity of S. Because $p^\perp \cap q^\perp$ is a polar space of rank at least 2. So, it is not singular space $m^\perp \cap (p^\perp \cap q^\perp)$. So let $w \in p^\perp \cap q^\perp \setminus m^\perp$. So we have a pentagon (x, p, w, q, m) with no diagonals.



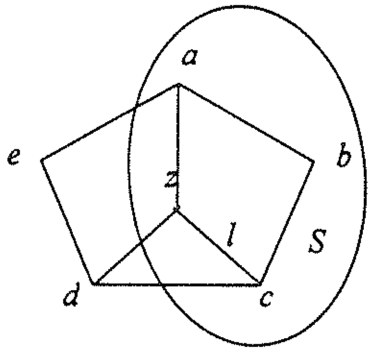
[Fig1]

Since Γ is a semi-strong, parapolar space then every pair of points of distance 2 in this pentagon is a polar pair, so $\langle x, q\rangle$ is a convex polar space (say S'). $S \cap S' \neq \varnothing$. Since $q \in S \cap S'$, it follows by hypotheses that $S \cap S'$ must contain a line, say l; x, l lie in the polar space S', then $x^\perp \cap l$ is a point z, say. $z \neq p$ since p is not collinear to p but $z \sim q$. This is a contradiction to the assumption that $x^\perp \cap S = \{p\}$. Thus $d(x, q) = 3$ as required whence (G) holds.■

The next two results can be found in [At1] however, again, these are proved for the wider class of semi-strong parapolar space rather than class of strong parapolar space

THEOREM 6.2. *Let* $\Gamma = (P, L)$ *be a semi-strong parapolar space. Then* (G) *is equivalent to* (wp).

PROOF. (G) $\Rightarrow$ (wp). If we have a pentagon with no diagonals and if it lies in a symplecton, then every vertex of the pentagon is collinear to a point on the opposite side and consequently (wp) hold. So, we may assume that (a, b, c, d, e) is a pentagon not in a symplecton. Then $\langle a, c\rangle$ is a symplecton because Γ is a semi-strong parapolar space S contains b but not d and e since otherwise the whole pentagon would



[Fig2]

lie in a symplecton against our assumption.

If $|d^{\perp}\cap S| = 1$ then by (G), (d, S) is a gated pair with gate c. But $d_{\Gamma}(a, d) \leq 2$ as can be seen in the figure. Now by (G) we have $2 \geq d_{\Gamma}(d, a) = 1 + d_S(c, a) = 1+2=3$ which is a contradiction. Therefore $d^{\perp}\cap S$ must contain a line l (say) on c as in the above figure. Since S is a polar space there is a point $z \in a^{\perp}\cap l \subseteq a^{\perp}\cap c^{\perp}\cap d^{\perp}$. Since (a, c, d) is an arbitrary non-consecutive triple of vertices from the pentagon whence (wp) holds

(wp) $\Rightarrow$ (G). Let (x, S) be a point-symplecton pair with $x^{\perp}\cap S = \{g\}$, it follows that

$$g^{\perp}\cap S \supseteq \Delta_2^{*}(x)\cap S$$

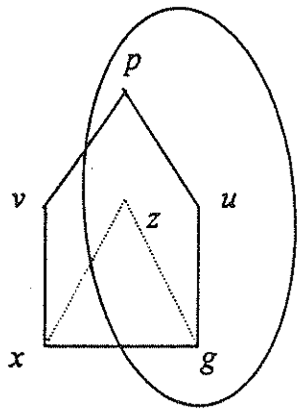
If (x, S) is not gated, then there is a point $p \in S\text{-}g^{\perp}$ of distance at most two from x. But as $x^{\perp}\cap S$ is a clique lying in $x^{\perp}$, x cannot be collinear with p. Therefor $d_{\Gamma}(x, p) = 2$. So, there exists $v \in x^{\perp}\cap p^{\perp}$.

Now $v \in (x^{\perp} \cap p^{\perp}) - g^{\perp}$, so v is not in S and one can choose $u \in p^{\perp} \cap g^{\perp}$ outside the clique $v^{\perp} \cap S$, since $p^{\perp} \cap g^{\perp}$ is a polar space of rank at least 2. So (p, u, g, x, v) is a pentagon with no diagonals. By (wp), there is a point z see the figure below such that

$$z \in p^{\perp} \cap g^{\perp} \cap x^{\perp}.$$

Since $S = \langle p, g \rangle$ is convex,

$$z \in x^{\perp} \cap S = \{g\}.$$



[Fig3]

This is impossible as $z \in p^{\perp}$ and $g \notin p^{\perp}$. Thus (x, S) is gated with gate g. This completes the proof.■

7 APPLICATIONS

A whole class of Lie incidence geometries is the subject of one and two theorems of Cohen-Cooperstein. This class satisfies the property that the rank of $x^{\perp} \cap S$ = -1, 1 or maximum rank in the following Theorem.

THEOREM 7.1. [COC] *Let Γ be a parapolar space and finite singular rank s. Then Γ satisfies:*

(a) *$x^{\perp} \cap S$ either empty, a line or a maximal singular subspace of S, for every point-symplecton pair with x not in S,*

(b) *$x^{\perp} \cap y^{\perp}$ has rank k.*

Iff there is a field F and one of the following holds:

(*i*) $k = s$, *Γ is a polar space of rank* $k + 1$.

(*ii*) $k = 3$, $s = 4$ *and* $\Gamma \cong D_{5,5}(F)$, $E_{6,4}(F)$ or $F_{4,1}(F)$.

(*iii*) $k = 4$, $s = 5, 6$ *and* $\Gamma \cong E_{6,1}(F)$, $E_{7,7}(F)$.

(*iv*) $k = 6$, $s = 7$ *and* $\Gamma \cong E_{8,1}(F)$.

Here we apply the above Theorem to show that all these geometries satisfy the weak pentagon property.

COROLLARY 7.2. *Let* $\Gamma = (P, L)$ *be a parapolar space with* $x^{\perp}\cap S$ *either empty, a line or a maximal singular subspace of* S. *Then* (wp) *holds in* Γ.

PROOF. Since $x^{\perp}\cap S$ is never a point then (G) holds in Γ. Therefore, by Theorem 6.2 (wp) holds in Γ.■

COROLLARY 7.3. (wp) *holds in the following geometries: polar spaces*, $D_{5,5}(F)$, $E_{6,4}(F)$, $F_{4,1}(F)$, $E_{6,1}(F)$, $E_{7,7}(F)$, *and* $E_{8,1}(F)$.

(Δ_2) **DEFINITION.** *In a parapolar space* $\Gamma = (P, L)$, *we say that* (Δ_2) *holds if for any point* $p \in P$, $\Delta_2^*(p)$ *is a subspace of* Γ.

PROPOSITION 7.4. *Let* Γ *be a semi-strong parapolar space, then* (wp) *implies* (Δ_2).

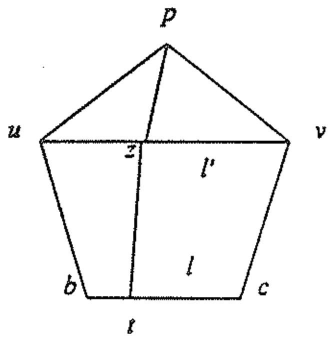
PROOF. (wp) $\Rightarrow$ (Δ_2). Suppose that a line l meets $\Delta_2^*(p)$ at two distinct point b, c clearly if $p^{\perp}\cap l \neq \varnothing$ then $\Delta_r^*(p)$ contains l. Also if p, b and c lie in a common symplecton then $l \subseteq \Delta_r^*(p)$.

So we may assume that no symplecton contains $\{p, b, c\}$ in particular $l\cap p^{\perp} = \varnothing$. Choose $u \in b^{\perp}\cap p^{\perp}$ if u is collinear with c, clearly $l \subseteq \Delta_r^*(p)$, so assume $u \notin c^{\perp}$, similarly choose $v \in c^{\perp}\cap p^{\perp}$, and if v is collinear with b, then $l \subseteq \Delta_r^*(p)$, so assume $v \notin b^{\perp}$.

If $v \sim u$, then since $\langle pu, pv\rangle$ is a projective plane having thick lines p is collinear with $l' = uv$, and since $|v^{\perp}\cap b^{\perp}| \geq 2$. Now since Γ is a parapolar space, $\langle v, b\rangle$ is a convex polar space by Cooperstein theory. Therefore every point t on the line bc, $t^{\perp}\cap l'$ is a point z. Then $z \in t^{\perp}$. p, z are both in the projective plane $\langle pv, vu\rangle$, then $z \sim p$ and thus $d(p, t) \leq 2$. It follows that every point t on the line be has at most distance 2 from p. Therefor $l \subset \Delta_r^*(p)$.

Finally, if u is not collinear with v, then (p, v, c, b, u) is a pentagon with no diagonal, but by (wp), so there exist a point w such that

$w \in p^{\perp} \cap b^{\perp} \cap c^{\perp}$



[Fig4]

This implies that $l \subseteq w^{\perp} \subseteq \Delta_r^*(p)$. This shows that in both cases $l \subseteq \Delta_r^*(p)$. Hence $\Delta_r^*(p)$ is a subspace.■

COROLLARY 7.5 (Δ_2) *holds in the following geometries: polar spaces,* $D_{5,5}(F)$, $E_{6,4}(F)$, $F_{4,1}(F)$, $E_{6,1}(F)$, $E_{7,7}(F)$, *and* $E_{8,1}(F)$.

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