PENTAGON PROPERTY IN PARAPOLAR SPACES II*

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ملخص البحث

في هذا البحث سنثبت أو ننفي تحقق " خاصية الشكل الخماسي "(*) في بعض الفراغات الهندسية البارقطبية و المنصوص عليها في نظريتي العالمين " كوهين و كوبرستين ".

ومن ثم نطبق هذه النتائج على مجموعة من بعض الهندسيات و خاصة هندسيات " لي " مثل الهندسيات:

$D_{6,6}(F), A_{5,3}(D), F_{4,1}(F), E_{7,1}(F)$

حيث F حقل، D حلقة.

(*) خاصية الشكل الخماسي: إذا كانت $ن_0$،$ن_1$، $ن_2$،$ن_3$،$ن_4$ شكل خماسي ليس له أقطار في الفراغ البارقطبي فإن هناك خطاً يصل بين النقطة $ن_ر$ و نقطة واحدة فقط في الخط المقابل $ن_{ر+2}$،$ن_{ر+3}$ حيث ر = 0،1،2،3،4 (الدلالات تؤخذ بمقياس 5).

ABSTRACT

In this thesis we prove or disprove that the "pentagon property"(*) holds in an important class of geometries) that are the subject of two theorems of Cohen and Cooperstein, by proving that the pentagon property is necessary to some other properties in different geometries.

We also apply these results to certain families of spaces. In particular, we apply the above result to the class of Lie incidence geometries: $D_{6,6}(F)$, $A_{5,3}(D)$, $F_{4,1}(F)$, $E_{7,1}(F)$ for some field F and division ring D.

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(*) Pentagon property:
If Γ =(P, L) is a point line geometry we say that pentagon property (P) holds in Γ if (x_0, x_1, x_2, x_3, x_4) is a pentagon with no diagonals then x_i is collinear to one point on the opposite line $x_{i+2}x_{i+3}$, with i = 0, 1, 2, 3,4 (indices are taken mod. 5).

In 1982, Arjeh Cohen wrote a fundamental and important paper [Co], in which he characterized the metasymplectic spaces $F_{4,1}(K)$ by only involving points and lines that can be considered " local characterization ", since those points and lines are within distance two of each other. He used an elementary property called "The pentagon property".

In 1989, Ernest Shult wrote a long paper [Sh1], in which he proved that the results, in [Co], can be proved by using a local version of the pentagon property.

In 1996, Mohammed El-Atrash, the co-author was studying the m-systems in parapolar spaces [At1], in which he needed the pentagon property to hold in those geometries he used, so he studied the pentagon property in various spaces [At2], [At3]. This paper is considered as a continuation of [At2], [At3].

1.GEOMETRIC PRELIMINARIES

1.1 BASIC DEFINITIONS

A point-line geometry Γ = (P, L) is a pair of sets; P is called the set of points and L is called the sets of lines, where the members of L are just subsets of P. If p is a point that belongs to a line l, we say that p lies on l or l passes through p or p is incident with l. A line is called ***thin*** if it contains exactly two points, otherwise it is called ***thick***.

In point-line geometry $\Gamma = (P, L)$, we say that two points $p,\ q \in$ P are ***collinear*** if they are incident with a common line, and we write $p \sim q$.

For any point p in point-line geometry $\Gamma = (P, L)$, define

$$p^{\perp} = \{p\} \cup \{q \in P \mid p \sim q\}.$$

For any set of points $X \subset P$, define $X^{\perp} = \cap\{p^{\perp} \mid p \in X\}$.

Rad(X) = $X \cap X^{\perp}$. In particular,
Rad(Γ) = $P^{\perp} = \{q \in P \mid p$ is collinear to q for all $p \in P\}$.

$\Gamma = (P, L)$ is called *a **linear*** (*or **singular***) space if each pair of distinct points lies exactly on one line. Γ is called *a **partial linear*** if each pair of points lies on at most one line.

*A **subspace*** of a point-line geometry $\Gamma = (P, L)$ is a subset X of points such that if a line l has at least two points of X then l lies entirely in X.

*A **path*** of length k from x_0 to x_k is a sequence of $k + 1$ points $x_0, x_1, \ldots, x_k$ such that x_{i-1} is collinear to x_i, $i = 1, 2, 3, \ldots, k$.

*A **geodesic*** is a shortest path between two points. We define the distance function

$$d_\Gamma : P \times P \to Z, \text{ by}$$
$$d_\Gamma(x, y) = \text{the length of any geodesic from } x \text{ to } y.$$

A subspace X is called ***convex*** if it contains all geodesics between any two points of X.
The smallest subspace containing a set X is called the ***subspace generated by*** X and denoted by ⟨X⟩.

A subspace X is called ***connected*** if for each pair of points there is a path that connects them and lies entirely in X.

The ***singular rank*** or just the ***rank*** of a space is the maximal number n for which there exists a chain of distinct subspaces

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n$$

such that X_i is singular for each i, with $X_i \neq X_j$, $i \neq j$. In this sense, the empty set has rank equals -1, rank($\{p\}$) = 0, for any point p, rank(l) = 1, for any line l.

Let p be a point in P, we define $\Delta_k(p)$ and $\Delta_k^*(p)$ as follows:

$$\Delta_k^*(p) = \{x \in P \mid x \text{ is of distance at most } k \text{ from } p\},$$

$$\Delta_k(p) = \{x \in P \mid x \text{ is of distance exactly } k \text{ from } p\}.$$

A ***geometric hyperplane*** is a subspace that meets every line of the space.

Let (P, L), (P', L') be two geometries. ***A morphism*** $\varphi: (P, L) \rightarrow (P', L')$ is a mapping between the indicated sets that preserves incidence and type, i.e., for any pair of points $x, y \in P$, we have

$\varphi(x), \varphi(y) \in P'$ and if:

$$x \sim y \Rightarrow \varphi(x) \sim \varphi(y)$$

Also if l is a line in L then $\varphi(l) \in L'$ and if $x \in l$ then $\varphi(x) \in \varphi(l)$

1.2 Some Families of Spaces

A point-line geometry is called ***projective plane*** if it satisfies the following conditions:

(أ) Γ is a linear space, i.e., every two distinct points $x, y \in P$ lie exactly on one line,

(ب) every two lines intersect in one point,

(ج) there are four points, no three of which are on one line.

A point-line geometry $\Gamma = (P, L)$ is called ***projective space*** if the following conditions are satisfied

(أ) every two points lie exactly on one line.

(ب) if l_1, l_2, are two intersecting lines then $\langle l_1, l_2 \rangle$ is a projective plane.

A ***gamma space*** is a point -line geometry such that for every point-line pair (p, l) with p not on l, p is collinear to no points of l, one point of l or all points of l.

It is easy to see that a gamma space is a space in which $p^{\perp}$ is a subspace, for every point $p \in P$.

In 1959, Veldkamp has published a paper [Ve], in which he axiomatically characterized certain families of geometries (Polar spaces) related to polarities in a vector spaces. In 1974, Buekenhout and Shult, in [BS], proved that the axiomatization can be dramatically simplified. They used the one-or-all axiom.

A ***polar space*** is a point-line geometry $\Gamma = (P, L)$ that satisfies the following Buekenhout-Shult one-or-all axiom

for each point p not incident with a line l, p is collinear with one point of l or all points of l.

It is easy to see that a polar space is a space in which $p^{\perp}$ is a geometric hyperplane, for every point $p \in P$.

A polar space $\Gamma = (P, L)$ is called ***degenerate*** if $rad(\Gamma) \neq \emptyset$, otherwise it is called ***non-degenerate***.

The ***polar rank*** or just the ***rank*** of a polar space is the maximal number n for which there exists a chain of distinct subspaces

$$rad(\Gamma) \subset X_1 \subset X_2 \subset \ldots \subset X_n$$

such that X_i is singular for each i, with $X_i \neq X_j$, $i \neq j$.

1.3 Classical Examples of Finite Polar Spaces

Let V be a vector space over a finite field F = GF(q), q is a prime power.

1. ***Symplectic Geometry*** $W_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a symplectic bilinear form B. In this case n is even, the polar space is of rank $n/2$.

2.***Hyperbolic Geometry*** $\Omega^+_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a hyperbolic bilinear form B. In this case n is even, the polar space is of rank $n/2$.

3. ***Elliptic Geometry*** $\Omega^-_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a elliptic bilinear form B. In this case n is even, the polar space is of rank $(n/2) - 1$.

4. ***Orthogonal Geometry*** $\Omega_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for an

orthogonal bilinear form B. In this case n is odd, the polar space is of rank
$(n\text{-}1)/2$.

5.*Hermitian Geometry* $H^{+}_{n}(q^{2})$ is the point-line geometry
(P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a Hermitian bilinear form B. In this case n is even, the polar space is of rank $n/2$.

6.*Hermitian Geometry* $H^{-}_{n}(q^{2})$ is the point-line geometry
(P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a Hermitian bilinear form B. In this case n is odd, the polar space is of rank
$(n\text{-}1)/2$.

Polar spaces of rank at least 3 have been classified in [BS], [B], [Jp], [BSP]. It turned out that those spaces in which lines contains at least 3 points and rank at least 4 are classical polar spaces.

A point-line geometry is called a ***parapolar space*** if it satisfies the following conditions.

(أ) Γ is a connected gamma space with no lines of cardinality two,

(ب) for every line; $l^{\perp}$ is not a singular space,

(ج) for every pair of points x, y of distance 2, $x^{\perp}\cap y^{\perp}$ is — either a point, or non-degenerate polar space of rank at least 2.

A parapolar space is called ***strong parapolar space*** if for every pair of points x, y of distance 2, $x^{\perp} \cap y^{\perp}$ is non-degenerate polar space of rank at least 2.

The pair of points (x, y) is called *a **polar pair*** if $x^{\perp} \cap y^{\perp}$ is non-degenerate polar space of rank at least 2. And the pair of points (x, y) is called *a **special pair*** if $x^{\perp} \cap y^{\perp}$ is a single point.

1.4 COOPERSTEIN'S THEORY

In [Coo], Cooperstein has shown that if (P, L) is a parapolar space, then for every pair of points p, q of distance 2, the convex closure $\langle p, q\rangle$ is a non-degenerate convex polar space whose rank is one more than the rank of $p^{\perp} \cap q^{\perp}$ and is independent of the choice of p, q. Moreover he has shown that all maximal singular subspaces of $\langle p, q\rangle$ are projective spaces. Such a convex polar space $\langle p, q\rangle$ is called a ***symplecton***. Convexity of such polar spaces forces the following very important property:

for any two distinct symplecta S_1, S_2 and a point p; $p^{\perp} \cap S_1$ and $S_1 \cap S_2$ are a projective subspaces of S_1(or S_2).

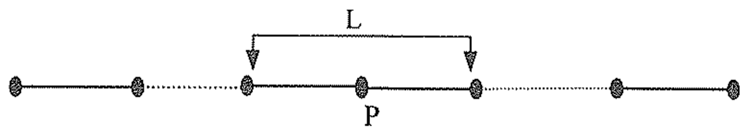
2. INTRODUCTION TO THE MAIN PROBLEM

2.1 LIE INCIDENCE GEOMETRIES

A point-line geometry $\Gamma = (P, L)$ whose points P are the cosets of a maximal parabolic subgroup of Lie type, and whose lines L are the cosets of the parabolic subgroup P' corresponding to the collection of all nodes in the dynkin diagram adjacent to the unique node corresponding to P, is called Lie incidence geometry. In this paper we mark the node of the Dynkin diagram that corresponds to the set of points by P, and we mark the node of the Dynkine diagram that corresponds to the set of lines by L. These geometries constitute a very important class of point-line geometries that have been under intensive study to investigate their properties and to characterize them using points and lines. In this study, we mainly apply almost all of our results to them.

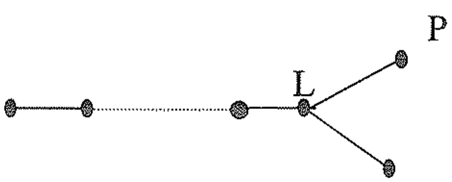
Next we will give some classical examples of finite spaces

$A_{n,k}(F)$



1 Grassmann Spaces

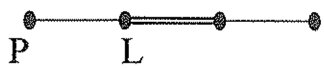
$\mathbf{D}_{n,n}(F)$



2 Half-spin Geometry

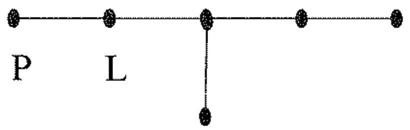
3 Metasymplectic spaces

$F_{4,1}(F)$

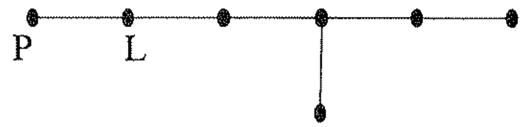


4 Exceptional Geometries

$E_{6,1}(F)$



$E_{7,1}(F)$



2.2 Properties of Parapolar Spaces

In a parapolar space $\Gamma = (P, L)$, if $x_0, x_1, x_2, x_3, x_4,$ are five point in P, we say that $(x_0, x_1, x_2, x_3, x_4)$ is a ***pentagon*** if x_i is collinear with x_{i+1}. In addition, we say that $(x_0, x_1, x_2, x_3, x_4)$ is a ***pentagon with no diagonals***, if $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon such that x_i is neither collinear with x_{i+2} nor collinear with x_{i+3} $i = 0, 1, 2, 3, 4$ (indices are taken mod 5).

Definition (2.2.1) (***Pentagon Property***) (P). *If* $(x_0, x_1, x_2, x_3, x_4)$ *is a pentagon with no diagonals then* x_i *is collinear to one point on the opposite line* $x_{i+2}x_{i+3}$ *with* $i = 0, 1, 2, 3, 4$ (*indices are taken mod* 5).

Definition (2.2.2) *A parapolar space* Γ *is called* ***semi-strong parapolar space*** *if in each pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *with no diagonals* $\langle x_i, x_{i+2}\rangle$ *is a symplecton (indices are taken mod 5) for all* $i = 0, 1, 2, 3, 4$. *In other words pentagons have no special pairs.*

This class was first introduced in [At3]. Strong parapolar space is semi-strong parapolar space. So semi-strong parapolar space is a class of parapolar spaces that lies in between parapolar spaces and strong parapolar spaces.
One sufficient condition for a parapolar space to be semi-strong parapolar space can be in the following result (see [At4]).

THEOREM (2.2.1) *If* Γ *is a parapolar space of rank* $r \geq 3$ *with* $x^{\perp} \cap S$ *is never a point for all point-symplecton pair* (x, S), $x \notin S$, *then pentagons with no diagonals have no special pairs.*

In a parapolar space $\Gamma = (P, L)$ if $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon with no diagonals, and if x_0 is collinear to a point on the opposite side x_2x_3 (say z) then we have $|x_0^{\perp} \cap x_3^{\perp}| \geq 2$. By Cooperstein Theory, $\langle x_0, x_3 \rangle$ is a symplecton (say S). Now S contains the two points z, x_3, and since S is a subspace it must contain the whole line x_2x_3, therefore $x_2 \in S$. Since S is a convex subspace then $x_1 \in S$. Hence the whole pentagon $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton.
On the other hand if a pentagon with no diagonals $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton S then because the space S is a polar space; x_0 is collinear to one point or all points of line x_2x_3. However x_0 can not be collinear to all points of x_2x_3 since the pentagon has no diagonals. Thus x_0 is collinear to one point on the other side. This argument will give us the following:
In a parapolar space, (p) is equivalent to saying that pentagons with no diagonals lie in a symplecton.

Some of the basic properties of polar spaces that are needed in this paper can be found in [Jp], [At3] and [Sh2]. Here we will state some of these results (without proof).

LEMMA (2.2.1) [Jp]. *Let* $\Gamma = (P, L)$ *be a polar space of rank*

≥ 2. *Then the following hold*:

(أ) *for any point* $p \in \mathrm{P}$, $l \in \mathrm{L}$, $p \in l$, *there is a point* $q \in \mathrm{P}$ *such that* $q^{\perp} \cap l = \{p\}$.

(ب) *if* X *is a subspace of* Γ *and if* $p \in \mathrm{P}$, *then* $p^{\perp} \cap \mathrm{X}$ *is either* X *or a hyperplane of* X.

LEMMA (2.2.2) [Sh2] *Non-degenerate polar space with thick lines is not the union of two singular subspaces.*

LEMMA (2.2.3) [At3] *Let* Γ *be a parapolar space. Let A, B be two distinct symplecta. If x, a, b are three points with* $a \in A \setminus B$, $b \in B \setminus A$, $x \in A \cap B$, $a \sim b$, *x is neither collinear to a nor to b. Then*

$$rank\,(a^{\perp} \cap B) = rank\,(A \cap B) = rank\,(b^{\perp} \cap A)$$

Here, we present the results which are obtained by El-Atrash and Cohen – Cooperstein (without proof)

Some similar results about (p) can be found in [At2] like the following result that finds equivalent condition for (p)

THEOREM [At2] *Let* Γ *be a parapolar space. Suppose that* $x^{\perp} \cap S$ *is empty or a line for all point-symplecton pair* (*x*, S), $x \notin S$. *Then the pentagon property holds if and only if* $S_1 \cap S_2$ *is never a line for every pair of distinct symplecta* S_1, S_2.

One of the geometries that satisfy these hypotheses is the metasymplectic spaces $F_{4,1}(K)$ for some field K.

Next are some results in various spaces. The first is a generalization of the previous result, the second relates (p) with its counterpart "The pentagon property holds locally". The third is a result that relates both (p) and (p) locally to the intersection of symplecta. All three results can be found in [At3].

THEOREM [At3] *Let* $\Gamma = (P, L)$ *be a parapolar space. Suppose that for some integer* $k \geq 1$; *rank* $(x^{\perp} \cap S) = -1$ *or* k *for every* $x \in P$ *and symplecton* S *with* $x \notin S$. *Then* (p) *holds in* Γ *iff rank* $(A \cap B) \neq k$; *for every pair of distinct symplecta* A, B.

THEOREM [At3] *Let* $\Gamma = (P, L)$ *be a parapolar space in which each symplecton has rank at least* 4. *Suppose that for some integer* $k \geq 1$; *rank* $(x^{\perp} \cap S) = -1$ *or* k *for every* $x \in P$ *and symplecton* S *with* $x \notin S$. *Then* (p) *holds in* Γ *iff* (p) *holds locally in* Γ.

THEOREM [At3] *Let* $\Gamma = (P, L)$ *be a parapolar space in which each symplecton has rank at least* 4. *Suppose that for some integer* $k \geq 1$; *rank* $(x^{\perp} \cap S) = -1$ *or* k *for every* $x \in P$ *and symplecton* S *with* $x \notin S$. *Then the following are equivalent:*

(i) (p) *holds in* Γ.

(ii) (p) *holds locally in* Γ.

(iii) *rank* $(A \cap B) \neq k$; *for every pair of distinct symplecta* A, B.

THEOREM [COC] *Let* $k \geq 2$ *and let* (P, L) *be a parapolar space with no thin lines, whose maximal singular subspaces have finite rank* s, *and whose symplecta have rank* $k + 1$. *Then* (P, L) *satisfies* $(x^{\perp} \cap S)$ *is either empty, a point, or a maximal singular subspace of* S *if and only if one of the following holds*:

(i) $k = s$ *and* (P, L) *is a non–degenerate polar space of rank* $k+1$ *with thick lines*;

(ii*a*) $k = 2, s \geq 3$, *and for some natural number* n *between* 4 *and* $2s$-1, *and a division ring* D, $(P, L) \equiv A_{n,d}(D)$, $d = n - s + 1$;

(ii*b*) $k = 2$, $s \geq 5$ *and* $(P, L) \equiv A_{2s-1,s}(D)/\langle\sigma\rangle$ *for some (infinite) division ring* D, *where* σ *is an automorphism of* $A_{2s-1,s}$ *induced by a polarity of the underlying projective space* PG($2s$-1, D) *of Witt index at most* s-5 ;

(iii) $k = 3$, $s \geq 4$ *and for some field* F, (P, L) *contains families* Σ, Π *of convex subspaces of* (P, L) *isomorphic to* $D_{4,1}(F)$ *and* $D_{5,5}(F)$ *respectively such that* Σ *is the system of symplecta of the parapolar space, and if* $(x, S) \in P \times \Sigma$ *with* $x \notin S$, $x^{\perp} \cap S$ *is a maximal singular subspace of S then* $\{x\} \cup S$ *lie in a unique member of* Π. *The incidence system of lines and planes lying on any point x is* $A_{s,2}(F)$.

(iv) $k = 4$, $s = 5$ *and* $(P, L) \equiv E_{6,1}(F)$ *for some field* F.

(v) $k = 5$, $s = 6$ *and* $(P, L) \doteq E_{7,1}(F)$ *for some field* F.

THEOREM [COC]. *Let* $k \geq 3$ *and suppose* (P, L) *is a gain a parapolar space with no thin lines, and finite singular rank s and whose symplecta have rank* $k + 1$. *Then* (P, L) *satisfies* $(x^{\perp} \cap S)$ *is either empty, a line, or a maximal singular subspace of S if and only if there exists a field* F *such that one of the following holds:*

(أ) $k = s$, (P, L) *is a non-degenerate polar space of rank* $k + 1$ *with thick lines*.

(ب) $k = 3$, $s = 4$ *and* $(P, L) \equiv D_{5,5}(F)$, $E_{6,4}(F)$ or $F_{4,1}(F)$.

(ج) $k = 4$, $s = 5$, 6 and $(P, L) \equiv E_{6,1}(F)$ or $E_{7,7}(F)$.

(د) $k = 6$, $s = 7$ *and* $(P, L) \equiv E_{8,1}(F)$.

3. MAIN RESULTS

This section is the care of the paper. For our study we need the following definition

DEFINITION. (Uniformazing Principle) For integer $k \geq 0$, we say that **(U_k)** holds in a parapolar space if for any two distinct symplecta R, S with rank$(R \cap S) \geq k$, we have for every $a \in R$, with $R \cap S \not\subset a^{\perp}$ implies that $a^{\perp} \cap S \not\subset R \cap S$.

The following is a necessary condition for (p) in parapolar spaces that satisfies (U_k) for $k = 2$.

LEMMA (3.1) *Let* Γ *be a parapolar space, satisfying* (U_2). *If* (P) *holds then rank* $(S_1 \cap S_2) < 2$ *for each distinct symplecta* S_1, S_2.

PROOF. Let Γ be a parapolar space as in the hypothesis. Suppose that (P) holds in Γ. Assume that rank $(S_1 \cap S_2) \geq 2$, for a pair of distinct symplecta S_1, S_2. Since S_1 is a polar space there is a point

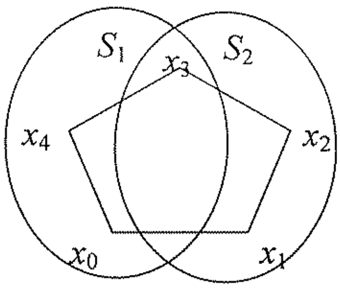


Fig. 1

$x_0 \in S_1 \backslash (S_1 \cap S_2)^{\perp}$. Let $x_3 \in S_1 \cap S_2$ with x_0 is not collinear to x_3. By (U_2) there is a point $x_1 \in (x_0^{\perp} \cap S_2) \backslash S_1$.
Since x_0 is not collinear to x_3, we can choose $x_4 \in x_0^{\perp} \cap x_3^{\perp}$, in fact we can choose x_4 not collinear to x_1 since $x_1^{\perp} \cap S_1$ is a singular subspace of S_1, however, $x_0^{\perp} \cap x_3^{\perp}$ is a polar space of rank at least 2, so it is not a singular subspace of S_1. Choose $x_2 \in (x_1^{\perp} \cap x_3^{\perp}) \backslash S_1$, again we can choose x_2 not collinear to x_4, since $x_2^{\perp} \cap S_1$ must be singular subspace of S_1, but $x_0^{\perp} \cap x_3^{\perp}$ is a polar space of rank at least 2. x_2 is not collinear to x_0, since $x_2^{\perp} \cap S_1$ is singular subspace, but x_0 not collinear to x_3. Similarly x_1 is not collinear to

x_4. Thus $(x_0, x_1, x_2, x_3, x_4)$ is pentagon with no diagonals. Then it follows by (p) that the pentagon $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton, say, S. Then by Cooperstein Theory $S_1 = S = S_2$. This contradicts our assumption that $S_1 \neq S_2$. Thus rank$(S_1 \cap S_2) < 2$, for each pair of distinct symplecta S_1, S_2. This completes the proof.(see Fig 1).

The next is in fact a generalization of the previous paper in [At5]. It is one of our main results since it involves those Lie incidence geometries that comprise the geometries of the first of two fundamental theorems of Cohen and Cooperstein.

THEOREM (3.1) *Let* $\Gamma = (P, L)$ *be a parapolar space, with the property that* $(x^{\perp} \cap S)$ *is either empty, a point, or a maximal singular subspace of* S*, for each point–symplecton pair* (x, S)*,* $x \notin S$*. If* (P) *holds in* Γ *then* $rank(S_1 \cap S_2) < 2$*, for each distinct symplecta* S_1*,* S_2*.*

PROOF: Assume that $rank(S_1 \cap S_2) \geq 2$, for some two

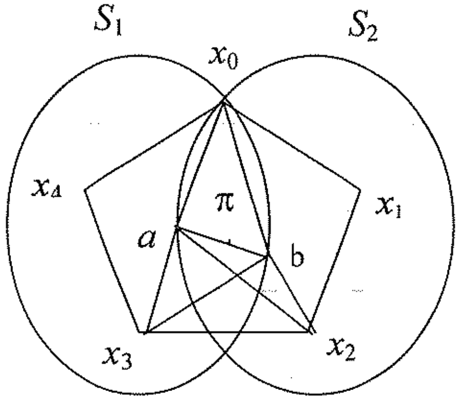


Fig. 2

distinct symplecta S_1, S_2. It follows that there is a plane say $\pi \subseteq (S_1 \cap S_2)$. Since S_1 is polar space we can find a point, say, $x_3 \in S_1$ such that $x_3^{\perp} \cap \pi$ is a line l $(= ab)$.
Let $x_0 \in \pi \backslash l$. Choose $x_4 \in x_0^{\perp} \cap x_3^{\perp}$. Since $l \subseteq x_3^{\perp} \cap S_2$, so $x_3^{\perp} \cap S_2$ is not empty and not a point, it follows from the hypothesis that $x_3^{\perp} \cap S_2$ must contain a maximal singular subspace of S_2 of rank $k \geq 2$. It follows that there is a point, say, $x_2 \in (x_3^{\perp} \cap S_2) \backslash l$. x_2 not collinear to x_0 since $x_2^{\perp} \cap S_1$ must be a singular subspace of S_1, however x_0 not collinear to x_3. Then we can choose $x_1 \in S_2 \cap x_0^{\perp} \cap x_2^{\perp}$. In fact we can choose x_1 not collinear to x_4 since $x_4^{\perp} \cap S_2$ must be singular subspace of S_2, however $x_0^{\perp} \cap x_2^{\perp}$ is not singular subspace of S_2. Similarly x_3 not collinear to x_1 and x_2 is not collinear to x_4. It follows that $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon with no diagonals. Then by (P) the pentagon $(x_0, x_1, x_2, x_3, x_4)$ must lie in a symplecton, say, S, however this implies that $S_1 = S = S_2$ by Cooperstein Theory. It contradicts our assumption that S_1, S_2 are distinct. This completes the proof.(see Fig 2).

Lemma (3.2) [At3] *Let* $\Gamma = (P, L)$ *be a parapolar space that is not a polar space in which there is some* $x \in P$ *and some symplecton* S_1 *with rank*$(x^{\perp} \cap S_1) = k$ *for some integer* $k \geq 2$. *Then there is a symplecton* S_2 *such that rank*$(S_1 \cap S_2) = k$.

Proof. Let S_1, x be as in the hypothesis, i.e., rank$(x^{\perp} \cap S_1) = k$, $k \geq 2$. Choose $y \in S_1$ such that $y^{\perp} \cap x^{\perp} \cap S_1$ is a hyperplane of $x^{\perp} \cap S_1$. Set $X = y^{\perp} \cap x^{\perp} \cap S_1$. Since $k \geq 2$, $S_2 = \langle x, y \rangle$ is a symplecton. We will show that rank $(S_1 \cap S_2) = k$. First we see that $\langle y, X \rangle$ is a singular subspace of rank k contained in $S_1 \cap S_2$.

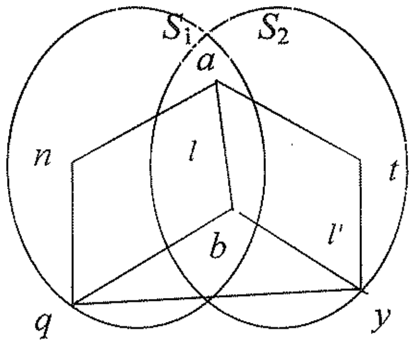
If $z \in S_1 \cap S_2$ then $z \in X^{\perp} \cap y^{\perp}$, that is, $z \in \langle X, y \rangle$. It follows that $S_1 \cap S_2 \subset \langle X, y \rangle$. Hence $S_1 \cap S_2 = \langle X, y \rangle$.

LEMMA (3.3) *Let Γ be a parapolar space in which $(x^{\perp} \cap S)$ is never a point for each point–symplecton pair (x, S), $x \notin S$. if* (p) *holds then $(S_1 \cap S_2)$ is never a line for each pair of distinct symplecta S_1, S_2.*

PROOF. Suppose that (p) holds in Γ. We want to show that rank$(S_1 \cap S_2) \neq 1$, for each pair of distinct symplecta S_1, S_2. Assume $S_1 \cap S_2$ is a line l, for some given two distinct symplecta S_1, S_2 in Γ. Since S_1 is a polar space we can find a point $q \in S_1 \backslash S_2$ such that $q^{\perp} \cap l$ is a point b. Let $a \in l$ with $a \neq b$. Since $a^{\perp} \cap q^{\perp}$ is a polar space of rank at least 2, we can choose $n \in a^{\perp} \cap q^{\perp}$ with $n \neq b$. Thus $q^{\perp} \cap S_2$ is not empty. It follows from hypothesis that there is a line, say, l' such that $l' \subseteq q^{\perp} \cap S_2$. Let $y \in l'$, $y \neq b$. y is not collinear to a since $y^{\perp} \cap S_1$ is singular space, however a not collinear to q. So we can choose $t \in (a^{\perp} \cap y^{\perp}) \backslash \{b\}$, in fact we can choose $t \notin n^{\perp}$, since $t^{\perp} \cap S_1$ must be singular subspace of S_1, however, $a^{\perp} \cap q^{\perp}$ is not a singular subspace. Similarly $n \notin y^{\perp}$, $q \notin t^{\perp}$. Thus (q, y, t, a, n) is a pentagon with no diagonals.
Since (p) holds in Γ then (q, y, t, a, n) lies in a symplecton (say S). This implies that $S_1 = S = S_2$ since $S = \langle a, q \rangle = S_1$, $S = \langle a, y \rangle = S_2$. This contradicts our assumption that S_1, S_2 are distinct symplecta.

This completes the proof.(see Fig 3).

Fig. 3



COROLLARY (3.1) *Let* Γ *be a parapolar space in which* $x^{\perp} \cap S$ *is either empty, line or maximal singular subspace of S, for each point–symplecton pair* (x, S), $x \notin S$. *If* (p) *holds then* $(S_1 \cap S_2)$ *is never a line for each pair of distinct symplecta* S_1, S_2

LEMMA (3.4) *Let* $\Gamma = (P, L)$ *be a parapolar space. Suppose that* $x^{\perp} \cap S$ *is either empty, line or singular subspace of rank k* ≥ 2 *for each point–symplecton pair* (x, S), $x \notin S$, *if* $(S_1 \cap S_2)$ *is never a line nor a singular subspace of rank k for every pair of distinct symplecta* S_1, S_2 *then* (p) *holds in* Γ.

PROOF: Let $(x_0, x_1, x_2, x_3, x_4)$ be a pentagon with no diagonals. Since $(x^{\perp} \cap S)$ is never a point for each point–symplecton pair (x, S), x $\notin$S, then by Theorem (2.2.1), Γ is semi-strong. Let $S_1 = \langle x_0, x_3 \rangle$, $S_2 = \langle x_0, x_2 \rangle$ be two symplecta. If $S_1 = S_2$, then the pentagon $(x_0, x_1, x_2, x_3, x_4)$ lies in a symplecton, and we are done. So assume $S_1 \neq S_2$. Since $x_2 \in x_3^{\perp} \cap S_2$. It follows from hypothesis that $(x_3^{\perp} \cap S_2)$ is either a line or a singular subspace of rank $k \geq 2$.

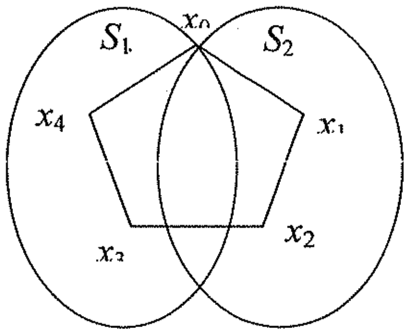


Fig. 4

Case 1. Rank($x_3^{\perp} \cap S_2$) = 1, then by the Rank Lemma

$$\text{rank } (S_1 \cap S_2) = \text{rank } (x_3^{\perp} \cap S_2) = 1.$$

Contradicting the hypothesis that ($S_1 \cap S_2$) is never a line nor a singular subspace of rank k

Case 2. Rank ($x_3^{\perp} \cap S_2$) = $k \geq 2$, then by the Rank Lemma

$$\text{rank}(S_1 \cap S_2) = \text{rank}(x_3^{\perp} \cap S) = k.$$

Contradicting the hypothesis that ($S_1 \cap S_2$) is never a line nor a singular subspace of rank k.(see Fig4).

LEMMA (3.5) *Let Γ be a parapolar space. Suppose that $x^{\perp} \cap S$ is either empty or contains a plane for each point–symplecton pair (x, S), $x \notin S$. Then $S_1 \cap S_2$ is never a line for each distinct symplecta S_1, S_2.*

PROOF. Suppose $S_1 \cap S_2$ is a line, say, l for a given two distinct symplecta S_1, S_2 in Γ. S_1 is a polar space therefore we can find $q \in S_1$ such that $q^{\perp} \cap l$ is a point b. Let $a \in l$, $a \neq b$. $\{b\} \subseteq q^{\perp} \cap S_2$, so $q^{\perp} \cap S_2 \neq \emptyset$. Then it follows from the hypothesis that $q^{\perp} \cap S_2 = k \geq 2$. Therefore we can choose $y \in S_2$ such that y collinear to q. y is not collinear to a since $y^{\perp} \cap S_1$ must be singular subspace of S_1, however q is not collinear to a.

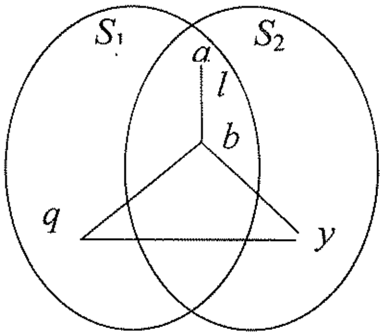


Fig. 5

By the Rank Lemma:

$2 \leq k = \text{rank}\,(q^{\perp} \cap S_2) = \text{rank}\,(S_1 \cap S_2) = 1.$
This is a contradiction. This completes the proof. (see figure 5).

4. APPLICATIONS

In this section we will state some of the properties of some Lie incidence geometries that are needed in the proofs of the next results [COC].

LEMMA (4.1) *Let Γ be the half-spin geometry* $D_{6,6}(F)$ *for arbitrary field F. Then for any point x and any pair of symplecta S, T we have:*

(i) $rank(x^{\perp} \cap S) = 0, 3, x \notin S.$

(ب) $rank(S \cap T) = -1, 1, 3$.

PROOF. For the proof see [COC].

COROLLARY (4.1) (p) *does not hold in* $D_{6,6}(F)$ *for arbitrary field F.*

PROOF. Since, by Lemma (4.1), there is a point x and a symplecton S with $x \notin S$ such that $rank(x^{\perp} \cap S) = 3$, it follows by Lemma (3.2) that there is a symplecton S_2 such that $rank(S_1 \cap S_2) = 3$, it follows then, by Theorem (3.1), that (p) does not hold in $D_{6,6}(F)$.

LEMMA (4.2) *Let Γ be the Grassmann geometry* $A_{5,3}(F)$ *for arbitrary field F. Then for any point x and any pair of symplecta S, T we have:*

(أ) $rank(x^{\perp} \cap S) = 0, 2, x \notin S$.

(ب) $rank(S \cap T) = -1, 1, 2$.

COROLLARY (4.2) (p) *does not hold in* $A_{5,3}(F)$ *for arbitrary field F.*

PROOF. Since, by Lemma (4.2), there is a point x and a symplecton S with $x \notin S$ such that $rank(x^{\perp} \cap S) = 2$, it follows by Lemma (3.2) that there is a symplecton S_2 such that $rank(S_1 \cap S_2) = 2$, it follows then, by Theorem (3.1), that (p) does not hold in $A_{5,3}(F)$.

LEMMA (4.3) *Let Γ be the geometry* $E_{7,1}(F)$ *for arbitrary field F. Then for any point x and any pair of symplecta S, T we have:*

(أ) $rank(x^{\perp} \cap S) = 0, 5, x \notin S$.

(ب) $rank(S \cap T) = -1, 1, 5$.

COROLLARY (4.3) (p) *does not hold in* $E_{7,1}(F)$ *for arbitrary field F.*

PROOF. Since, by Lemma (4.3), there is a point x and a symplecton S with $x \notin S$ such that $rank(x^{\perp} \cap S) = 5$, it follows by Lemma (3.2) that there is a symplecton S_2 such that $rank(S_1 \cap S_2) = 5$, it follows then, by Theorem (3.1), that (p) does not hold in $E_{7,1}(F)$.

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