Properties of the geometry of type $D_{5,2}(F)$

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ملخص البحث

قدمنا في هذا البحث نوعا جديدا" من هندسة نقطة–خط من نوع $D_{n,2}(F)$. حيث أننا قد عرضنا بالتفصيل تركيب وتعريف هذه الهندسة الجديدة وخير مثال لهذا النوع من الهندسة هو عندما ن=5. يتناول الجزء الأساسي من هذا البحث العديد من الخواص ولا سيما أهم هذه الخواص التي قمنا بإثباتها وهي أن هذه الهندسة باراقطبية وليست قوية وأثبتنا أن قطرها يساوي 3 ، كما أثبتنا أيضا أن خاصية الشكل الخماسي متحققة في هذه الهندسة.

Abstract

In this paper we have introduced a kind of point-line geometry of type $D_{n,2}(F)$. Many subjects have been discussed in detail such as the definition and the construction of this new geometry, a good example of this geometry is when n equals 5. The main part of this work has investigated many properties of this geometry. The most important properties which have been proved in this paper is that $D_{5,2}(F)$ is not strong parapolar space of diameter three. In addition we have proved as well that the pentagon property (p) is satisfied in the geometry of type $D_{5,2}(F)$.

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1- Introduction

Let V be a vector space of finite dimension n over an arbitrary field F. ***A bilinear form*** B on V is a mapping $B : V \times V \to K$, such that for $\alpha, \beta \in K$; $x, y, z \in V$ we have:

(i) $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$.

(ii) $B(z, \alpha x + \beta y) = \alpha B(z, x) + \beta B(z, y)$.

Thus a bilinear form is a linear functional in each of its coordinate.

A vector $u \in V$ is called ***an isotropic*** vector, if $B(u, u) = 0$, and a subspace $W \subseteq V$ is called ***totally isotropic*** *subspace* of *V* if and only if $B(u, v)=0$ for all $u, v \in W$.

Given a set I, *a geometry* Γ over I is an ordered triple $\Gamma=(X, *, D)$, where X is a set, D is a partition $\{X_i\}$ of X indexed by I, X_i are called components, and $*$ is a symmetric and reflexive relation on X called incidence relation such that: $x * y$ implies that either x and y belong to distinct components of the partition of X or $x = y$. Elements of X are called *objects* of the geometry, and the objects within one component X_i of the partition are called *the objects of type i*. The subscripts that index the components are called *types*. The obvious mapping

τ: X→I, which takes each object to the index of the component of the partition containing it is called the *type map* τ.

A *point-line geometry* (P, L) is simply a geometry for which |I |=2, one of the two types is called *points*; in this notation the points are the members of P, and the other type is called *lines*. Lines are the members of L. If $p \in P$ and $l \in L$, then $p * l$ *if and only if* $p \in l$. In a point-line geometry (P, L), we say that two points of P are *collinear* if and only if they are incident with a common line. (We use the symbol ~ for collinear)

$x^{\perp}$ means the set of all points in P collinear with x, including *x* itself.

A subspace of a point-line geometry Γ=(P, L) is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. <X> means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called *thick* lines, those incident with exactly two points are called *thin* lines.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there exist a chain of distinct subspaces

$\emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$, for example rank($\emptyset$)=-1, rank($\{p\}$) = 0 where p is a point and rank(L) = 1 where L a line.

In a point-line geometry Γ=(P, L), a path of length n is a sequence of n+1 ($x_0, x_1, .., x_n$) where, (x_i, x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. ***A geodesic*** from a point x to a point y is a path of minimal possible length with initial point *x* and end point *y*. We denote this length by d_Γ (x, y).

A geometry Γ is called ***connected*** if and only if for any two of its points there is a path connecting them. A subset X of P is said to be ***convex*** if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry Γ=(P, L) satisfying the Buekenhout-Shult axiom **[SH]** :

For each point-line pair (*p*, *l*) with *p* not incident with *l*; *p* is collinear with one or all points of *l*, that is $|p^\perp \cap l| = 1$ or else $p^\perp \supset l$. Clearly this axiom is equivalent to saying that $p^\perp$ is a geometric hyperplane of Γ for every point $p \in$P.

A point-line geometry Γ=(P, L) is called ***a projective plane*** if and only if it satisfies the following conditions **[PJ]** :

(i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of them are on a line.

A point-line geometry Γ=(P, L) is called ***a projective space*** if the following conditions are satisfied:

(i) every two points lie exactly on one line ,

(ii) if l_1, l_2 are two lines $l_1 \cap l_2 \neq \emptyset$, then $\langle l_1, l_2 \rangle$ is a projective plane. ($\langle l_1, l_2 \rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

A point-line geometry Γ=(P, L) is called ***a parapolar*** *space* if and only if it satisfies the following properties:

(i) Γ is a connected gamma space,

(ii) for every line l; $l^{\perp}$s not a singular subspace,

(iii) for every pair of non-collinear points x, y; $x^{\perp} \cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If x, y are distinct points in P, and if $|x^{\perp} \cap y^{\perp}|=1$, then (x, y) is called ***a special pair***, and if $x^{\perp} \cap y^{\perp}$ is a polar space, then (x, y) is called ***a polar pair*** (or ***a symplectic pair***). A parapolar space is called ***a strong parapolar*** space if it has no special pairs **[CC1]**.

In this part we shall define the varieties of the geometry $\mathbf{D}_{n,2}$ **(F)** in which we have found out that the symplecta of such geometry are of some kind of the Grassmannian. So it is convenient here to look at some of their relevant properties. Proofs can be found in [**CC2, CC1**]. We give the definition and construction geometry of type $\mathbf{D}_{n,2}$ **(F)** in Sec. 3, while the properties of the geometry of type $\mathbf{D}_{5,2}$ **(F)** are discussed in Sec. 4. Finally, the pentagon and weak pentagon properties are explaned in Sec. 6.

2- GRASSMANNIANS

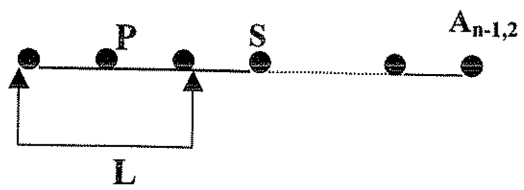
2.1. Proposition:[At1,CC1] Let $\Gamma = A_{m,n}(D)$, $2 \leq n \leq 2m-1, m \geq 2$

i- $\Delta^*_{m-1}(p)$ is a hyperplane of Γ for any point p, if and only if m=2n-1.

ii- If $x, y \in P$ and $d(x, y)=d$, then the convex closure $\langle x, y \rangle$ is isomorphic to $A_{2d-1,d}(D)$, $d \geq 2$.

2.2. Proposition [CC2]. Let (P, L)= $A_{n,2}(D)$ for $n \geq 4$ and some division ring D. Then we have the following:

i- (P, L) has the following diagram.



ii- (P, L) is a strong parapolar space of diameter 2.

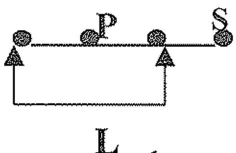
iii- (P, L) satisfies $(\mathbf{P3})_2$ and $(\mathbf{F4})_{\{-1\}}$.

iv- If S,T$\in$ $\mathbb{S}$ and S$\neq$T, then rank (S$\cap$T)=-1,0,2.

v- We have rank (S$\cap$T)=0,2, for all distinct S, T$\in$ S, if and only if $n \leq 5$.

2.3. Proposition [CC2]. Let (P, L)= $A_{4,2}(D)$ for some division ring D. Then we have the following:

ii- (P, L) has the following diagram.



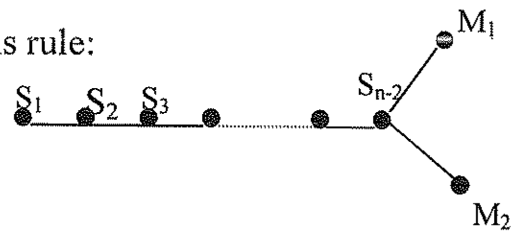
ii- (P, L) is a strong parapolar space of diameter 2.

iii- (P, L) satisfies $(\mathbf{P3})_2$ and $(\mathbf{F4})_{\{-1\}}$.

iv- If S,T$\in$ $\mathbb{S}$ and S$\neq$T, then rank (S$\cap$T)=2.

3- Definition of the geometry of type $D_{n,2}$ (F)

Let B be a non-degenerate orthogonal hyperbolic symmetric form on a vector space V of even dimension 2n over a finite field *F of order k*. Let S_I be the set of all totally isotropic *i*-dimensional subspaces of *V;* $1 \leq i \leq$n-2. Let S_n be the class of all maximal totally isotropic subspaces of dimension *n*. S_n is partitioned into two classes denoted by M_1 and M_2 subject to this rule:

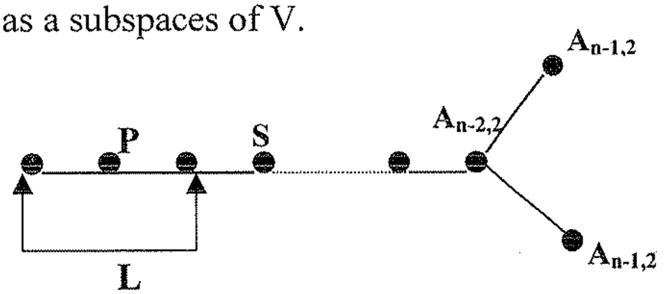


n. S_n is partitioned into two classes denoted by M_1 and M_2 subject to this rule:

Two maximal totally isotropic subspaces M_1 and M_2 of V belong to the same class if and only if $M_1 \cap M_2$ has the same parity as n, *i.e.* if n is odd, then M_1, M_2 belong to the same class iff $M_1 \cap M_2$ is of dimension1,3,5,….,n, and if n is even, then M_1, M_2 belong to the same class iff $M_1 \cap M_2$ is of dimension 0,2,4,……,n.

The geometry of type $\mathbf{D_{n,2}}$ **(F)** is the point-line geometry (P, L), whose set of points P is the collection of all T.I (totally isotropic) 2-dimensional subspaces of the vector space V, and whose lines are the pairs (A,B) where A is a T.I 1-dimensional subspace of the T.I 3-dimensional subspace B – that is, the set of (1,3)-subspace flags.

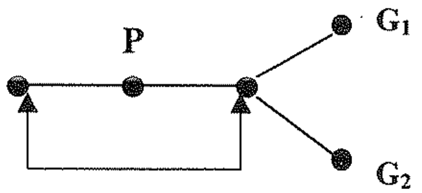
A point C is incident with a line (A, B) if and only if $A \subset C \subset B$ as a subspaces of V.

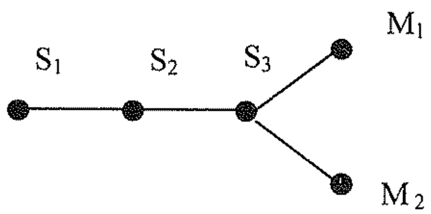


Construction of $\mathbb{D}_{5,2}$ (F). Consider the polar space $\Delta = \Omega^{+}(10, F)$ that comes from a vector space of dimension 10 over a finite field F= GF(k) with a symmetric hyperbolic bilinear form. The two classes M_1, M_2 consist of maximal totally isotropic 5-dimensional subspaces. Two 5-subspaces fall in the same class if their intersection is of odd dimension. So the dimension of the intersection of $M_1 \neq M_2$ is 1, 3, or 5.

The geometry of type $\mathbb{D}_{n,2}$ (F) is the point-line geometry (P, L), whose set of points P is the collection of all 2-dimensional subspaces of the vector space V, and whose lines are the pairs (A,B) where A is a 1-dimensional subspace of the 3-subspace B – that is, the set of (1,3)-subspace flags. A point C is incident with a line (A, B) if and only if $A \subset C \subset B$ as a subspaces of V.

To define the collinearity, let C_1 and C_2 be two point (the points are the T.I 2-spaces), then C_1 is collinear to C_2 if and only if the intersection of $C_1 \cap C_2$ =1-space and $\langle C_1, C_2 \rangle$=3-space.





Polar space Δ

The elements of the classes $\mathbf{G_1}$ and $\mathbf{G_2}$ are geometries of type $A_{4,2}(F)$ and we define them as:

$\mathbf{G_1}$, $\mathbf{G_2}$ $=\{ A_N |\ A_N$ (Geometry of type $A_{4,2}(F)$) is the set of all T.I 2-spaces that is contained in N, $N \in M_1$ or $M_2\}$

4- Properties of the geometry of type $\mathbf{D_{5,2}}$ (F)

$\mathbf{D_{5,2}}$ **(F)** *is a connected non-degenerate not strong parapolar space of diameter three.*

4.1 Proposition. *The geometry* (P, L)= $\mathbf{D_{5,2}}$ **(F)** *is of diameter* 3.

Proof: Suppose that p and q are two arbitrary points in P. Then we have 2-spaces $\Psi(p)=C_1=<x_1,x_3>$ and $\Psi(q)=C_2=<x_2,x_4>$. If $<x_1,x_3>\cap<x_2,x_4>$ is a 1-space, say $<x>$, where $x=x_1=x_2$, then we have two cases:

1- $x_3^{\perp}\cap C_2=C_2$

2- $x_3^{\perp}\cap C_2=<x>$.

In case 1, $<x_3,x_4,x>$ forms a 3-space and $(<x>,<x,x_3,x_4>)$ is a line containing the two points, where $<x>\subseteq<x,x_3>$, $<x,x_4>\subseteq<x,x_3,x_4>$. Thus the two points are collinear, this means that d(p,q)=1.

In case 2, $B(x_3,x_4)\neq 0$ and C_2 is contained in a 5-space T, say $T=\langle x,x_4,u,v,w\rangle$, meanwhile x_3 is outside T. Then $x_3^{\perp}\cap T$ is either T or a

hyperplane of T the first is impossible; because the maximal TI space is 5-space, so $x_3^{\perp}\cap T$ is a hyperplane of T that is $\langle x,u,v,w\rangle$. Then we have two lines $(\langle x\rangle,\langle x,x_3,u\rangle)$ and $(\langle x\rangle,\langle x,x_4,u\rangle)$ which means that the 2-space $\langle x,u\rangle$ is the third point that is collinear to each of C_1 and C_2. Thus $d(p,q)=2$.

If $\langle x_1,x_3\rangle\cap\langle x_2,x_4\rangle\}$=0-space, then we have three cases:

1- $x_1^{\perp}\cap C_2=C_2$

2- $x_3^{\perp}\cap C_2=C_2$

3- $x_1^{\perp}\cap C_2=\langle x_2\rangle$ and $x_3^{\perp}\cap C_2=\langle x_4\rangle$

In case 1, $\langle x_2,x_3,x_4\rangle$ and $\langle x_2,x_3,x_1\rangle$ form 3-spaces. Thus $(\langle x_3\rangle,\langle x_2,x_3,x_4\rangle)$ and $(\langle x_3\rangle,\langle x_2,x_3,x_1\rangle)$ are lines containing the points $\langle x_1,x_4\rangle$, $\langle x_2,x_4\rangle$ and $\langle x_2,x_3\rangle$. Then we have: $\langle x_3\rangle\subseteq\langle x_2,x_4\rangle$, $\langle x_3,x_1\rangle\}\subseteq\langle x_1,x_2,x_3\rangle$ and $\langle x_2\rangle\subseteq\langle x_2,x_3\rangle$, $\langle x_2,x_4\rangle\}\subseteq\langle x_2,x_3,x_4\rangle$. This means that $d(p,q)=2$.

Case 2 is similar to case 1.

In case 3, we have a 5-space containing $\langle x_2,x_4\rangle$, say $T=\langle x_2,x_3,x,y,z\rangle$, then $x_3^{\perp}\cap T=\langle x_4,x,y,z\rangle$ and $x_1^{\perp}\cap T=\langle x_2,x,y,z\rangle$. Thus $\langle x_1,x_3,x,y,z\rangle$ forms a 5-space, then

$<x,x_3,x_1>$, $<x,x_3,x_4>$ and $<x,x_4,x_2>$ are 3-spaces. Then we have:$<x_3>\subseteq<x_1,x_3>$, $<x_3,x>\subseteq<x_1,x_3,x>\}$, $<x>\subseteq<x_3,x>$, $<x,x_4>\subseteq<x,x_3,x_4>$ and $<x_4>\subseteq<x,x_4>$, $<x_2,x_4>\subseteq<x,x_2,x_4>\}$. This means that we have three lines: ($<x_3>$,$<x_1,x_3,x>$) is the first line containing the points $<x_1,x_3>$ and $<x,x_3>$. ($<x>$,$<x_3,x_4,x>$) is the second line containing the points $<x,x_3>$ and $<x,x_4>$. ($<x_4>$,$<x_2,x_4,x>$) is the third line containing the points $<x,x_4>$ and $<x_2,x_4>$. Hence d(p,q)=3. □

4.2. Corollary. (P, L)= **$D_{5,2}$ (F)** *is a partial linear connected space.*

Proof. For any two distinct points C_1, C_2 (the points are totally isotropic 2-dimensional subspaces of V) we have either $C_1\cap C_2=\varphi$ then we can not find any T. I.(totally isotropic) 3-space including 1-space and contains both of them, so, the points are not collinear or from the previous discussion, the collinear two points $C_1=\langle x_1,x\rangle$ and $C_2=\langle x_2,x\rangle$ have only one T.I 3-space that contains each of $C_1=\langle x_1,x\rangle$ and $C_2=\langle x_2,x\rangle$, which is $\langle x, x_1, x_2\rangle$. Thus every two points have at most one line.

4.3. Corollary. The geometry (P, L)= **$D_{5,2}$ (F)** is a gamma space.

4.4. Proposition. $D_{5,2}$ (F) *is a parapolar space.*

Proof. The geometry is a connected gamma space has bee done. For any line l we show that $l^{\perp}$ is not singular. Assume that l have two points p and q, where $\Psi(p)=<x_2,x>$ and $\Psi(q)=<x_1,x>$, then $l=(<x>,<x,x_1,x_2>)$. If we take r to be a point such that $\Psi(r)=<x_1,x_2>$, then r is not incident to l and $(<x_1>,<x,x_1,x_2>$ is containing the points r and p, $(<x_2>,<x,x_1,x_2>$ is a line containing the points r and q, then $r\in l^{\perp}$. Since the point $<x,x_2>$ is contained in a 5-space, say $T=<x,x_2,u,v,w>$, then $x_1^{\perp}\cap T=<x,x_2,u,v>$. Take the point s to be the 2-space $<w,x>$, then $(<x>,<x,x_1,w>)$ is a line containing the points p, s and $(<x>,<x,x_2,w>)$ is a line containing the points q, s. then $s\in l^{\perp}$. Since $<x_1,x_2>\cap<x,w>$=0-space, ris not collinear to s. Hence $l^{\perp}$ is not singular. Proposition 4.1 and Cor. 4.2 automatically satisfy the remaining part of the conditions. □

4.5. Proposition. (P, L)= $\mathbf{D}_{5,2}$ **(F)** *is not strong space.*

Proof. Consider the case of the two points of the form $C_1=<x_1,x_3>$, $C_2=<x_2,x_4>$ such that $x_3^{\perp}\cap C_2=C_2$, and $x_1^{\perp}\cap C_2=<x_2>$, then $B(x_1,x_4)=1$. If

$Z=<x_3,x_2>$ is a 2-space that falls in the intersection of two 3-spaces containing $C_1=<x_1,x_3>$, $C_2=<x_2,x_4>$ respectively, then Z, $<x_2,x_4>$ lie in a 3-space and they intersect in a 1-space, say x. Since $x_1\sim x$, then x must be x_2 and same for Z, $<x_1,x_3>$ their intersection is a 1-space, say y. So $y\in<x_1,x_3>$ but $x_4\sim y$, then y must be x_3. Thus the only 2-space is $<x_3,x_2>$. Therefore the points C_1 and C_2 are special. □

5- Symplecta of $D_{5,2}$ (F) and their properties

There are two kinds of symplecta in $D_{5,2}$ (F) the first is the Grassmannians of type $A_{3,2}(F)$ that are located as symplecta of the geometries of type $A_{4,2}(F)$. Since each object of the classes **G₁** and **G₂** are a Grassmannians of type $A_{4,2}(F)$ and each of them corresponds to a T.I 5-dimensional subspaces of V, they are classified into the two classes **G₁** and **G₂** according to the rule:

Two geometries A_{M_1} and A_{M_2} (each of them is a geometry of type $A_{4,2}$) belong to different classes if and only if $M_1\cap M_2$ is of dimension 0, 2 or 4 (M_1 and M_2 are the correspondent maximal T.I 5-dimensional subspaces to A_{M_1} and A_{M_2} respectively).

Remark 1. We shall give a notation for each geometry of type $A_{4,2}$ in **$D_{5,2}$ (F)** such as A_T which means that the set of all T.I. 2-spaces are contained in the T.I. 5-space T.

Remark 2. Each geometry of type $A_{3,2}(F)$ will be denoted by A_D which means that the set of all T.I. 2-spaces are contained in the T.I. 4-space D.

So we say a geometry $A_{3,2}(F)$ is a symplecton of a geometry $A_{4,2}$ and then a symplecton of the geometry **$D_{5,2}$ (F)** iff A_D is a symplecton of A_T and $D \subseteq T$.

A_{M_1} and A_{M_2} belong to the same class if and only if $M_1 \cap M_2$ is of dimension 1, 3 or 5.

Now the symplecta of the geometry **$D_{5,2}$ (F)** are the geometries of type $A_{3,2}(F)$ that are corresponding to the set of all T.I. 4-spaces.

For example A_D (A_D is a geometry of type $A_{3,2}(F)$ and D is a T.I 4-space) is a symplecton of $D_{5,2}$ (F) iff there exist a T.I 5-space (corresponding to the geometry $A_{4,2}$ in which $A_{3,2}(F)$ is a symplecton) containing D, we define symbolically the class of symplecta (S)of this kind as:

S={A_D| A_D is the set of all T.I. 2-space that contained in D and $D \subseteq T$, $T \in M_1$ or M_2}.

The second kind of symplecta is a geometry of type D_4, in which each symplecton corresponds to a T.I 1-dimensional subspace of V and we define this kind as: $D_4=\{D(x) \mid D(x)$ is the set of all T.I 2-space that contains x, $x \in S_1\}$.

5.1. Proposition. i- *Let* A_{D_1}, A_{D_2} *be two symplecta with* $A_{D_1} \neq A_{D_2}$, *then* rank($A_{D_1} \cap A_{D_2}$)=-1, 0, 2.

ii- If *symplecta* D_1 *and* D_2 *are* of *type* D_4, *then* rank($D_1 \cap D_2$)=-1, 0.

Proof. i- If A_{D_1} and A_{D_2} are symplecta of the same geometry A_T, then $D_1 \subseteq T$ and $D_2 \subseteq T$, so $D_1 \cap D_2 =$T.I. 3-space which means that rank($A_{D_1} \cap A_{D_2}$)=2.

Now if A_{D_1} and A_{D_2} are symplecta of different geometries A_{T_1} and A_{T_2} respectively, then we have two cases for A_{T_1} and A_{T_2}:

1- A_{T_1} and A_{T_2} are in the same class (The classes G_1 and G_2). In this case we have either dim($T_1 \cap T_2$)=1 or 3, for dim($T_1 \cap T_2$)=1, if we choose $D_1 \cap D_2$ = the same 1-space, then there is no any T.I. 2-space contained in each of D_1 and D_2 this gives $A_{D_1} \cap A_{D_2}$=φ and rank($A_{D_1} \cap A_{D_2}$)=-1. For dim($T_1 \cap T_2$)=3, we have three possibility $D_1 \cap D_2$ = 1-space or 2- space or 3-space, this gives the following: $A_{D_1} \cap A_{D_2}$=φ or a point or a plane respectively so, rank($A_{D_1} \cap A_{D_2}$)=-1, 0, 2.

2- A_{T_1} and A_{T_2} are located in different classes (one of them in the class G_1 and the other in the class G_2), then we have dim($T_1 \cap T_2$)=0, 2, 4. For dim($T_1 \cap T_2$)=0, any choice of D_1 and D_2 gives $D_1 \cap D_2$ = φ and $A_{D_1} \cap A_{D_2}$=φ, then rank($A_{D_1} \cap A_{D_2}$)=-1. For dim($T_1 \cap T_2$)=2, one choices of $D_1 \cap D_2$ = 2-space, then $A_{D_1} \cap A_{D_2}$ is a single point and rank($A_{D_1} \cap A_{D_2}$)=0. The last case where dim($T_1 \cap T_2$)=4 has only two choices, either $D_1 \cap D_2$=2-space or 3-space, which means that $A_{D_1} \cap A_{D_2}$ is a single point or a plane, then rank($A_{D_1} \cap A_{D_2}$)=0, 2. Hence we deduce from the all previous cases that rank($A_{D_1} \cap A_{D_2}$)=-1, 0, 2.

ii- For this kind of this symplecta we have exactly two cases either the T.I. 1-spaces x(D_1) and x(D_2) are disjoint or

$\langle x(D_1),x(D_2)\rangle$ is a T.I. 2-space. Thus $D_1 \cap D_2$ is either empty or a point and so rank($D_1 \cap D_2$)=-1, 0.

Remarks. We have proved in **$D_{5,2}$ (F)** that if A_{D_1} and A_{D_2} are two symplecta of the same geometry A_T (A_T is a Grassmannian geometry of type $A_{4,2}$(F)), then rank($A_{D_1} \cap A_{D_2}$)=2. This result agrees with the assertion of proposition **2.3. [CC2]**.

5.2. Proposition. *Let (p, A_D) be a pair of a point p and a symplecton A_D, then either $p^{\perp} \cap A_D$ is empty or a projective plane.*

Proof. There are two cases either $p \in A_T$ or $p \notin A_T$. If $p \in A_T$, then the T.I. 2-space *C* (the point *p*) is contained in the T.I. 5-space *T*. Then we have either the case that the T.I. 4-space *D* ($D \subseteq T$) meets *C* is in a 1-space say x, then the set of all T.I. 2-spaces that are in *D* and contain x constitute a T.I. 3-space (a projective plane). Thus $p^{\perp} \cap A_D$ is aprojective plane i.e., rank($p^{\perp} \cap A_D$)=2 or $C \subseteq D$ which means that $p \in A_D$. If $p \notin A_T$, then there are two cases:

1- $C \cap T$=1-space, say x, then either $C \cap D$=x, so the set of all T.I. 2 spaces that are containing x constitute 3-space in D. At the same time we have $\langle C - x\rangle^{\perp} \cap T = D$, then $p^{\perp} \cap A_{D}$ is a plane i.e., rank($p^{\perp} \cap A_{D}$)=2 or $C \cap D$=φ and $p^{\perp} \cap A_{D}$=φ, thus rank($p^{\perp} \cap A_{D}$)=-1

2- $C \cap T$=φ, in this case $C \cap D$=φ so, there is no any T.I. 2-space in each of C and D i.e., $p^{\perp} \cap A_{D}$=φ, thus rank($p^{\perp} \cap A_{D}$)=-1. Hence all cases give us two possibilities either rank($p^{\perp} \cap A_{D}$)=-1 or 2.

5.3. Proposition. *If (p, D) is a non-incidence pair of point p and a symplecton D of type* D_4, *then either $p^{\perp} \cap D$.is a line or empty.*

Proof. the pair of (p, D) is a pair of T.I. 2-space C and a T.I. 1-space $x(D)$, then either $x^{\perp} \cap C$ is a 1-space or all C. If $x^{\perp} \cap C$ is 1-space, then there is no any T.I. 2-space that is containing the 1-space and collinear to p which means $p^{\perp} \cap D$=φ. If $x^{\perp} \cap C$=C, then there are two different T.I. 2-spaces contain $x(D)$ and collinear to the point p, this means that there are two collinear points (a line) in D and each of them is collinear to p. Hence $p^{\perp} \cap D$ is a line and rank($p^{\perp} \cap D$)=-1, 1.

5.4. Proposition. If p and q are two distinct point of the geometry $\mathbf{D}_{5,2}(\mathbf{F})$ with d(p, q)=2, then the convex closure $\langle p,q \rangle$ is either a symplecton of type D_4 or a symplecton of type $A_{3,2}$.

Proof. Since p and q are non-collinear points then by proposition 4.1, either $C_1 \cap C_2$=1-space (the points p and q are the T.I. 2-spaces C_1 and C_2 respectively) or C_1 and C_2 constitute a T.I.4-space D. If $C_1 \cap C_2$=1-space, then the convex closure $\langle p,q \rangle$ is a symplecton of type D_4. The second case gives a symplecton of type A_D.

6- Pentagon property

El Atrash in [**At1**] has given an equivalent property to **(p)** in parapolar spaces of rank at least three in which we shall show that the *pentagon property hold in the geometry* $\mathbf{D}_{5,2}(\mathbf{F})$.

6.1. Definition (pentagon property, **[At2]**).

(P) If x_0, x_1, x_2, x_3, x_4 are five points in a parapolar space, where x_i is collinear to x_{i+1} and x_i is not collinear to x_{i+2} then x_i is collinear to one point on the line $x_{i+2}x_{i+3}$ (indices are taken modulo 5).

We say that $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon with no diagonals if x_i is neither collinear to x_{i+2} nor x_{i+3} (indices are taken modulo 5).

6.2.Theorem [At1]. Let Γ be a parapolar space of rank at least three. Suppose that $x^{\perp} \cap S$ is empty or a line for all point-symplecton pair (x,S), $x \notin S$. Then the pentagon property holds if and only if $S_1 \cap S_2$ is never a line for every pair of symplecta S_1, S_2.

6.3. Corollary (P) *holds in the geometry* of *type* $\mathbf{D_{5,2}}$ **(F).**

Proof. In $\mathbf{D_{5,2}}$ **(F)** for each pair of point-symplecton (x, S) and $x \notin S$ we have proved in propositions 5.2 and 5.3 that $x^{\perp} \cap S$ is either empty or a line. Since we proved also in proposition 5.1 that $S_1 \cap S_2$ empty or a point or a plane for every pair of symplecta S_1, S_2, by Theorem 6.2 the pentagon property holds in $\mathbf{D_{5,2}}$ **(F).**

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