®-Compact Operators Between Locally Convex Spaces

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ملخص البحث

قمنا في هذا العمل بدراسة بعض أنواع الفئات شبه المحكمة (التي تسمى محكمة-(ر)) التي متتابعاتها من الاقطار النونية تؤول إلى الصفر بمعدلات مختلفة وعرفنا العلاقة $c_®$ حيث $(E,F) \in c_®$ إذا وفقط إذا كان كل مؤثر خطي متصل منE إلي F يكون محكم-(ر) وبرهنا أن كل مؤثر خطي متصل من فراغ معيارى E إلي أي فراغ محدب محليا F يكون محكم-(ر) إذا وفقط إذا كانت كل فئة محددة في F تكون محكمة-(ر).

Abstract

In this work we study some type of Precompact sets (which called ®-compact) whose sequences of the nth diameters converge to zero in different rates (rapidly, radically,...) and we define the relation $c_®$ where $(E,F) \in c_®$ if and only if every continuous linear operator T from E into F is ®-compact and we prove that every continuous linear operator from a normed space E into a locally convex space F is ®-compact if and only if every bounded subset of F is ®-compact.

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1. ®-Compact Sets.

Introduction. For any normed space E, the bounded subset D of E is called precompact if and only if

$$(\delta_n(D, B_E))_{n=0}^{\infty} \in c_0, \qquad (1)$$

here B_E is the closed unit ball in E.

In [1] Astala and Ramanvjan define (s)-nuclear (resp. $\Lambda(\alpha)$-nuclear) sets in normed space by replacing c_0 in (1) by the space (s) of rapidly decreasing sequences (resp. the power series space $\Lambda(\alpha)$).

In [2] Faried, and Ramadan define ®-compact sets in normed space by replacing c_0 in (1) by any sequence ideal.

In [8] Zahriuita studies the relation $\boldsymbol{R}$ between locally convex spaces, where $(E,F) \in \boldsymbol{R}$ if and only if every continuous linear operator $T : E \to F$ is compact.

In this paper we give the necessary and sufficient conditions for $(E,F) \in c_{®}$.

By c_0 we denote the space of all sequences of real numbers that converge to zero.

By (S), we denote the space of all rapidly decreasing sequences of real numbers given by:

$$(S) = \left\{ (\lambda_n)_{n=1}^{\infty} : \sup_n n^{\alpha} |\lambda_n| < \infty \ \forall \alpha > 0 \right\}.$$

By $\Lambda(\alpha)$, $\alpha=(\alpha_n)_{n=1}^{\infty}$, $0<\alpha_1\leq \alpha_2\leq\ldots;\ \alpha_{2n}\leq c\,\alpha_n$, for some $c>0$, we denote the power series space of all sequences of real numbers given by:

$$\Lambda(\alpha)=\left\{(\lambda_n)_{n=1}^{\infty}:\ \sup_n R^{\alpha_n}|\lambda_n|<\infty\ \forall R>0\right\}.$$

By (R), we denote the space of all radical sequences of real numbers given by:

$$(R)=\left\{(\lambda_n)_{n=1}^{\infty}:\ \lim_n \sqrt[n]{|\lambda_n|}=0\right\}$$

By $[x]$ we mean the integer part of real number x such that $[x]=\alpha$ if $x=\alpha+\beta, 0\leq\beta<1$.

Remark: Each radical sequence is rapidly decreasing and the converse is not necessarily true.

In fact, if we take $\lambda_n=\frac{1}{2^n}, n\in N$, then $\lim_n \lambda_n n^{\alpha}=\lim_n \frac{n^{\alpha}}{2^n}=0$ for all $\alpha>0$. But since $\sqrt[n]{\frac{1}{2^n}}=\frac{1}{2}$, we have $(\lambda_n)_{n=1}^{\infty}\in(S)\setminus(R)$ (see [2], proposition 1).

Definition 1.1. [2] A sequence ideal ® on a scalar field is a subset of the space l_∞ (the space of all bounded sequences of real numbers) satisfying the following conditions:

(i) $e_i\in$ ®, where $e_i=(0,0,\ldots,1,\ldots)$ the one in the ith place.

(ii) If $x_1,x_2\in$ ®, then $x_1+x_2\in$ ®.

(iii) If $y \in l_\infty$ and $x \in$ ®, then $x.y \in$ ®.

(iv) If the sequence $x = (x_0, x_1, ...) \in$ ®, then $(x_{\left[\frac{n}{2}\right]})_{n=0}^{\infty} = (x_0, x_0, x_1, x_1, ...) \in$ ®.

Note that the sequence spaces c_0 , (S), (R) and $\Lambda(\alpha)$ are examples of sequence ideals (see[2], page 9).

Definition 1.2. Let A, D be absolutely convex sets in a topological vector space E such that D absorbs A, i.e. there exists $\lambda > 0$ such that $A \subset \rho D$ for all $\rho > \lambda$. For a subspace F of E we define:

$$\delta(A, D; F) = \inf\{r > 0 : A \subset rD + F\}.$$

the nth diameter of A with respect to D is defined as

$$\delta_n(A, D) = \inf\{\delta(A, D; F) : \dim(F) \le n\}, \quad n = 0,1,2,.....$$

it satisfies the following properties :

(1) $\delta_0(A, D) \ge \delta_1(A, D) \ge ... \ge \delta_n(A, D)... \ge 0.$

(2) $\delta_n(A, D) = 0$ if and only if A is contained in a linear subspace of E of dimension at most n.

(3) If A is a bounded subset of E, then A is precompact if and only if

$$\lim_n(\delta_n(A, U)) = 0 \quad \forall U \in \mu(E),$$

where $\mu(E)$ is a local base of zero in E.

(4) If $T : E \to F$ is a linear operator , then

$$\delta_n(T(A),T(D)) \le \delta_n(A,D).$$

(5) If $A_1 \subset A$ and $D \subset D_1$, then $\delta_n(A_1,D_1) \le \delta_n(A,D)$.

(6) If $\alpha,\beta > 0$, then $\frac{\alpha}{\beta}\delta_n(A,D) = \delta_n(\alpha A,\beta D)$.

Definition 1.3. For a sequence ideal $® \subset c_0$, a subset D of a locally convex space E is called ®-compact if and only if $(\delta_n(D,U))_{n=0}^{\infty} \in ®$ for all $U \in \mu(E)$.

Note that any bounded subset D of a normed space E is ®-compact if and only if $(\delta_n(D,B_E))_{n=0}^{\infty} \in ®$, where B_E is the closed unit ball in E.

For examples:

(1) If $\dim(E) = n$, then every bounded subset of E is ®-compact.

(2) If $D = \left\{x = (x_n) : \sum_{n=1}^{\infty}|x_n|2^n \le 1\right\}$ and $B = \left\{x = (x_n) : \sum_{n=1}^{\infty}|x_n|n \le 1\right\}$ are two subsets of l_1, then according to [5], 9.1.3 we have $\delta_n(D,B_{l_1}) = \frac{1}{2^n}$ and $\delta_n(B,B_{l_1}) = \frac{1}{n}$, where B_{l_1} is the closed unit ball in l_1. Hence D is precompact and rapidly-compact, but not radically compact and B is precompact but not rapidly compact .

Proposition 1.4. If D_1, D_2 are ®-compact subsets of locally convex space E, then $D_1 + D_2$ is ®- compact .

Proof. Since D_1 and D_2 are ®- compact sets, then

$$(\delta_n(D_1,U))_{n=0}^{\infty} \in ® \text{ and } (\delta_n(D_2,U))_{n=0}^{\infty} \in ® \ \forall U \in \mu(E),$$

hence

$$(\delta_{\left[\frac{n}{2}\right]}(D_1,U))_{n=0}^{\infty} \in ® \text{ and } (\delta_{\left[\frac{n}{2}\right]}(D_2,U))_{n=0}^{\infty} \in ® \ \forall U \in \mu(E).$$

If $W \in \mu(E)$, then there exists a neighborhood U of zero in F such that $U + U \subset W$. According to the definitions of $\delta_n(D_1,U)$ and $\delta_m(D_2,U)$ we have for any $\varepsilon > 0$ that there exist subspaces F_1 and F_2 of F with $\dim(F_1) \le n$ and $\dim(F_2) \le m$ such that

$$D_1 \subset (\delta_n(D_1,U) + \varepsilon)U + F_1,$$

$$D_2 \subset (\delta_m(D_2,U) + \varepsilon)U + F_2.$$

Hence we have

$$D_1 + D_2 \subset (\max(\delta_n(D_1,U), \delta_m(D_2,U)) + \varepsilon)(U + U) + (F_1 + F_2) \subset$$
$$(\max(\delta_n(D_1,U), \delta_m(D_2,U)) + \varepsilon)W + (F_1 + F_2).$$

Since $\dim(F_1 + F_2) \le n + m$, then

$$\delta_{n+m}(D_1 + D_2, W) \le (\max(\delta_n(D_1,U), \delta_m(D_2,U)) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\delta_{n+m}(D_1 + D_2, W) \le \max(\delta_n(D_1,U), \delta_m(D_2,U)).$$

Hence for all $W \in \mu(E)$, we have

$$\delta_r((D_1+D_2),W) \le \max(\delta_{\left[\frac{r}{2}\right]}(D_1,U),\delta_{\left[\frac{r}{2}\right]}(D_2,U)) \le$$

$$\delta_{\left[\frac{r}{2}\right]}(D_1,U)+\delta_{\left[\frac{r}{2}\right]}(D_2,U),\ \ r \in N.$$

and therefore $(\delta_r(D_1+D_2),W))_{r=0}^{\infty} \in$®, thus D_1+D_2 is ®-compact subset of E.

2. ®-Compact operators.

Definition 2.1. For a sequence ideal ®$\subset c_0$ and two locally convex spaces E and F, an operator T from E into F is called ®-compact operator if and only if there exists a neighborhood V of zero in E such that $(\delta_n(T(V),U))_{n=0}^{\infty} \in$® for all $U \in \mu(F)$.

Definition 2.2.

We say that an order pair of locally convex spaces (E,F) satisfies the condition c® (and we write $(E,F) \in$ c®) if every continuous linear operator $T: E \to F$ is ®-compact.

Lemma 2.3. Let E_j, $j=1,2,...,n$, F be locally convex spaces and $E=\prod_{j=1}^{n} E_j$. Then

(i) from $(E_j,F) \in$ c®, $j=1,2,...,n$, it follows that $(E,F) \in$ c®,

(ii) from $(F,E_j) \in$ c®, $j=1,2,...,n$, it follows that $(F,E) \in$ c®.

Proof.

(i) Let $T: E=\prod_{j=1}^{n} E_j \rightarrow F$ be an arbitrary continuous linear operator. Then the operator $H_j: E_j \rightarrow F$ defined by $H_j(x_j)=T(0,\ldots,x_j,\ldots,0)$ is continuous linear operator. Since $(E_j,F)\in$ c®, then H_j is ®- compact operator, hence there exists a neighborhood U_j of zero in E_j such that $H_j(U_j)$ is ®-compact set in F. Since $T(U_1\times\ldots\times U_n)=$ $T(U_1\times\{0_{E_2}\}\times\ldots\times\{0_{E_n}\})+\ldots+$ $T(\{0_{E_1}\}\times\ldots\times\{0_{E_{n-1}}\}\times U_n)=H_1(U_1)+\ldots+H_n(U_n)$, then by proposition (1.4) we have T is ®- compact operator.

(ii) Let $T: F\rightarrow E=\prod_{j=1}^{n} E_j$ be any arbitrary linear continuous operator. Since the projection map $P_i: E=\prod_{j=1}^{n} E_j \rightarrow E_i$ is continuous, then $P_i oT$ is a continuous operator from F to E_i. Since $(F,E_i)\in$ c®, it follows that $P_i oT$ is ®-compact operator, hence there exists a neighborhood U of zero in F such that $P_i oT(U)=W_i$ is ®-compact set in E_i. It follows by [5], theorem (2.6) that $W_1\times W_2\times\ldots\times W_n=T(U)$ is ®-compact set in E, hence T is ®-compact operator.

Definition 2.4. A locally convex space E is said to be of type (s®), if for every neighborhood U of zero in E there exists a neighborhood V of zero in E which is ®-compact with respect to U, (i.e.,

$(\delta_n(V,W))_{n=0}^{\infty} \in$ ® for all neighborhood of zero $W \subset U$).

Proposition 2.4. Suppose E is a locally bounded and locally convex space. Then E is a space of Type (s®) if and only if $(E,F) \in$ C® for every normed space F.

Proof. Sufficiency. Let $(E,F) \in$ C® for every normed space F. Since E is locally bounded and locally convex, then it has a bounded convex neighborhood U of zero such that the collection $\mu(E)=\left\{\frac{1}{n}U : n \in N\right\}$ is a local base of zero in E. For U we can define a semi-norm P_U such that

$$P_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}, \quad (x \in E).$$

We shall associate for U a semi-normed space E_U which is E with the semi-norm P_U. Let

$$E_U / N_U = \left\{\hat{x} = x + N_U : x \in E\right\},$$

where $N_U = \{x \in E : P_U(x) = 0\}$, be the quotient space. If $\hat{P}_U(\hat{x}) = P_U(x)$, then by [7], page 31, $\hat{P}_U$ is a norm of the quotient space E_U / N_U, hence E_U / N_U is a normed space.

Next, the quotient map $\pi_U : E \to E_U$ defined by

$$\pi_U(x) = x + N_U$$

is continuous. Since E_U is a normed space, and since $(E, E_U) \in C_{®}$, then π_U is ®-compact operator. Thus, there exists a neighborhood V of zero in E such that $\pi_U(V)$ is ®-compact set in E_U, so $(\delta_n(\pi_U(V), \pi_U(U))_{n=0}^{\infty} \in ®$. By [4], page 209, we have $\delta_n(V,U) = \delta_n(\pi_U(V), \pi_U(U))$, hence $(\delta_n(V,U))_{n=0}^{\infty} \in ®$.

Now for every neighborhood H of zero in E, let $W \subset H$ be any neighborhood of zero in E, since $\mu(E)$ is a local base of zero in E, then there exists $n \in N$ such that $\frac{1}{n}U \subset W$, it follows that

$$\delta_n(V,W) \le \delta_n(V, \frac{1}{n}U) \le n\delta_n(V,U),$$

hence $(\delta_n(V,W))_{n=0}^{\infty} \in ®$, so V is ®-compact with respect to H, thus E is a space of type $(S_{®})$.

Necessity. Let E be a space of type $(S_{®})$, F be an arbitrary normed space and $T : E \to F$ be an arbitrary linear continuous operator. Since F is a normed space, the topology on F defined by the norm $\|\cdot\|$. Let $B_F = \{x \in F : \|x\| \le 1\}$ be the unit ball in F, which is a neighborhood of zero in F. Since T is continuous, there exists a neighborhood U of zero in E such that $T(U) \subset B_F$. By assumption, E is a space of type ($S_{®}$), so there exists a neighborhood

V=V(U) of zero in E, which is ®-compact with respect to U i.e. $(\delta_n(V,W))_{n=0}^{\infty} \in ®$ for all $W \subset U$. But since $\delta_n(TV,TW) \le \delta_n(V,W)$, we have $(\delta_n(TV,TW))_{n=0}^{\infty} \in ®$. Since also $T(W) \subset T(U) \subset B_F$, then $\delta_n(TV,B_F) \le \delta_n(TV,TW)$. Thus, T is ®-compact operator. So it is proved that $(E,F) \in C_{®}$ for every normed space F.

3. ®-Montel Spaces

A locally convex space E is called barrelled if every absolutely convex absorbent closed set in E is a neighborhood.

Definition 3.1 A locally convex space E is called ®-Montel, if it is barrelled and every bounded subset D of E is ®-compact.

Note that every finite dimensional normed space is ®-Montel.

Proposition 3.2 A necessary and sufficient condition for a locally convex space F to be ®- Montel is $(E,F) \in C_{®}$ for every normed space E.

Proof. Sufficiency. Let F be an ®-Montel space, E be a normed space, and $T : E \to F$ be a continuous linear operator.

Since E is a normed space, then the topology on E defined by the norm $\|.\|$. If $B_E = \{x \in E : \|x\| \leq 1\}$ is the closed unit ball in E, then T maps the unit ball B_E in E into a bounded set $T(B_E)$ in F. Since E is ®-Montel space, $T(B_E)$ is ®-compact set in F (because in ®-Montel spaces bounded sets and ®-compact sets coincide). Hence, T is ®-compact operator, which shows that $(E,F) \in$ C®.

Necessity, suppose $(E,F) \in$ C® for every normed space E. We shall show that every bounded set A in F is ®-compact. Since F is a locally convex space, it has a local base β of zero such that each $U \in \beta$ is absolutely convex and absorbent. Let $\beta = \{U_\lambda : \lambda \in L\}$, where L is an index set. Then to each absolutely convex absorbent set U_λ, we can define a corresponding semi-norm $\|.\|_\lambda$ such that

$$\|y\|_\lambda = \inf\{\alpha : \alpha > 0, y \in \alpha U_\lambda\},$$

which is the Minkowski functional for U_λ. Since U_λ is a neighborhood of zero, then $\|.\|_\lambda$ is continuous. Thus, the set $Q = \{\|.\|_\lambda : \lambda \in L\}$ of continuous semi-norms defines the topology τ of F. Let $K = \{V_{\lambda,\varepsilon} : \varepsilon > 0, \lambda \in L\}$ where

$$V_{\lambda,\varepsilon} = \{y \in F : \|y\|_\lambda \leq \varepsilon\} \quad (\varepsilon > 0, \lambda \in L),$$

so a local base μ of zero for this topology τ is formed by the sets,

$$V = \bigcap_{i=1}^{n} V_{\lambda_i,\varepsilon} \quad (V_{\lambda_i,\varepsilon} \in K).$$

Now, by the definition of a bounded set, since A is bounded set in F, then $\forall V \in \mu \quad \exists s = s_V > 0$, such that $A \subset s_V V$. It follows that $\forall \lambda \in L\ \exists s(\lambda) > 0$ such that

$$A \subset s(\lambda) V_{\lambda,1} = s(\lambda)\{y \in F : \|y\|_\lambda \le 1\}.$$

If $y \in A$, then $y = s(\lambda)w$, for some $w \in V_{\lambda,1}$. Hence,

$\|y\|_\lambda = s(\lambda)\|w\|_\lambda \le s(\lambda)$, so $\dfrac{\| y \|_\lambda}{s(\lambda)} \le 1$. Put $m(\lambda) = \dfrac{1}{s(\lambda)}$ >0, then $\forall \lambda \in L\ \exists m(\lambda) > 0$ such that

$$p(y) = \sup\{m(\lambda)\|y\|_\lambda : \lambda \in L\} \le 1 \quad (y \in A);$$

which is a semi-norm on A. If $p(y) = 0$, then $m(\lambda)\|y\|_\lambda = 0$ for all $\lambda \in L$, hence $\|y\|_\lambda = 0$ for all $\lambda \in L$. Since Q is a separating family of semi-norms, then $y = 0$, hence P is a norm. Let $E = \{y \in F : p(y) < \infty\}$, then $E \subset F$ is a normed space. Let $B_E = \{y \in F : p(y) \le 1\}$ be the unit ball of E, then A $\subseteq B_E$. Let the operator T equal to the identity imbedding of E into F. Since for all $\lambda \in L$ there exists $c(\lambda) = \dfrac{1}{m(\lambda)}$ such that

$$\|Ty\|_\lambda = \|y\|_\lambda \le c(\lambda) p(y),$$

then T is continuous. Thus, since $(E,F) \in C_{®}$, then T is ®-compact operator. Therefore $B_E = T(B_E)$ is ®-compact set in F. But since $A \subset B_E$, then A is ®-compact set in F, hence F is ®-Montel space.

Proposition 3.3 If E is a barrelled locally convex space of type $(S_{®})$, then E is ®-Montel space .

Proof. Let A be a bounded set in E, then for each neighborhood U of zero in E there exists $t_U > 0$ such that $A \subseteq rU$ for every $r \geq t_U$. Since E is a space of type $(S_{®})$, there exists a neighborhood V of zero in E such that for all neighborhood $W \subset U$, we have

$(\delta_n(V,W))_{n=0}^{\infty} \in ®$, so there exists $t_V > 0$ such that $A \subseteq sV$ for every $s \geq t_V$. It follows that

$$\frac{1}{s}\delta_n(A,U) \leq \frac{1}{s}\delta_n(A,W) \leq \delta_n(\frac{1}{s}A,W) \leq \delta_n(V,W),$$

hence $(\delta_n(A,U))_{n=0}^{\infty} \in ®$ for all $U \in \mu(E)$, so A is ®-compact, thus E is ®-Montel space.

Example (Finite Convex Topology). Any vector space E can be made into a convex space by taking as a base of neighborhoods of the zero the set of all absolutely convex absorbent subsets.

This is the finest convex topology on E. By [6], page 75, every bounded sets in E is finite dimensional, so every bounded sets in E is ®-compact. Hence E is ®-Montel space.

References

[1] Astala, k. and Ramanvjan, M.S. "(s) Nuclear Sets and Operators". Pacific Journal of Mathematics, Vol. 127, No. 2, (1987).

[2] Faried, N. and Ramadan, F. " N-Diameters and Kolmogrove Numbers of Finite Cartesian Product of ®-Compact sets and Direct Sums of Operators." Proc. Math. Phys. Soc. Egypt, No. 68, (1992), 1-10.

[3] Faried, N., Sarsour, j. and Safi Z., ®-compact sets and operators", al Aqsa. (to appear).

[4] Jarchow, H. "Locally Convex Spaces." B. G. Teubner Stuttgart, (1981).

[5] Pietch, A. " Nuclear Locally Convex Spaces", Akademic Verlin, Berlin, (1972).

[6] Pietch, A. "'Operator Ideals", North-Holland Publishing Company, Amsterdam, New York, oxford, (1980).

[7] Rudin, W., "Functional Analysis", McGraw-Hill, New York, (1973).

[8] Zahriuita, V.P."On The Isomorphism of Cartesian Product of Locally Convex Spaces ". Studia Math. 46, (1973), 201-221.