Peristaltic Flow Through a Porous Medium in a Tube

By

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ملخص البحث

يختـــبر هـــذا البحث الأنسياب التمعجي لمائع نيوتوني غير قابل للإنضغاط خلال وسط مسامي في أنـــبوبة أسطوانية شبه متماثلة. وقد استخدمت متسلسلة الاضطراب (للرتبة الأولى) بدلالة العدد الموجي غير البعدي لموجة لانهائية وذلك للحصول على صورة تحليلية صريحة لكل من السرعة وتدرج الضغط. أيضـــاً فقد ناقشنا تأثير بارامتر النفاذية على كل من ضغط الامتلاء وقوة الاحتكاك. وتشير النتائج العددية إلى أن ضغط الامتلاء يزداد كلما تناقص معامل النفاذية.

Abstract

The paper investigates the peristaltic flow of an incompressible Newtonian fluid through a porous medium in axisymetric cylindrical tube. A perturbation series (to first order) in dimensionless wave number of an infinite harmonic travelling wave is used to obtain an explicit form for the velocity and the pressure gradient. Also, we discuss the effect of permeability parameter on both pressure rise and friction force. The numerical results show that the pressure rise increases as the permeability decreases.

KEYWORDS: Peristalsis, Newtonian Fluid, Cylindrical Tube, Pporous Medium

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1. Introduction

Peristaltic transport has been the subject of many recent studies because of its importance in many engineering and medical applications. After the study of Latham [4], several investigations have been made to understand peristaltic action in both mechanical and physiological situations. A review of much of the early literature is presented in an article by Jaffrin and Shapiro [3]. A summary of most of the experimental and theoretical investigations reported, with details of the geometry, fluid, Reynolds number, wavelength parameter and wave amplitude parameter has been given in a paper by Srivastava and Srivastava [8].

Because of its importance to geophysicists, physicists, engineers and mathematicians, the fluid motion through a porous medium have been studied by a number of authors, such as Varshney [10], Raptis *et al.* [5,7], Raptis and Peridikis [6], and El-Dabe and El-Mohandis [2].

Recently, Sobh [9] investigate peristaltic flow of a Newtonian fluid through a planar channel filled with a homogenous porous medium. Using a perturbation expansion on the wave number as a parameter, he obtained analytical form for the velocity field and the pressure gradient.

Since glandular ducts and other tracts of the body in which peristalsis occur are approximately cylindrical in shape, the axisymmetric case is of particular interest to physiologists. Accordingly, we purpose to investigate the peristaltic flow through a porous medium in cylindrical tube. Using a perturbation expansion with the wave number as a parameter, the velocity field and pressure gradient are obtained in explicit form. Moreover, the pressure rise per unit wavelength and the friction force are computed numerically and graphed versus flow rate for various values of physical parameters.

2. Formulation and Analysis

We shall consider the flow of an incompressible Newtonian viscous fluid through a homogenous porous medium in an axisymmetric flexible cylindrical tube with travelling sinusoidal waves of moderate amplitude imposed on its wall. The geometry of the wall surface is

$$\bar{h}(\bar{Z},\bar{t}) = d + b\sin\frac{2\pi}{\lambda}(\bar{Z} - c\bar{t}), \qquad (2.1)$$

where d is the radius of the tube, b is the wave amplitude, λ is the wavelength and $\bar{t}$ is the time.

We choose the cylindrical coordinate system $(\bar{R},\bar{Z})$, where the $\bar{Z}$-axis lies along the centerline of the tube, and $\bar{R}$ is the distance measured radially. Let $\bar{U},\bar{W}$ be the velocity components in the radial and the axial directions respectively. In the laboratory frame $(\bar{R},\bar{Z})$, the flow in the tube is unsteady but if we choose moving coordinates $(\bar{r},\bar{z})$ which travel in the $\bar{Z}$-direction with the same speed as the wave (wave frame), then the flow can be treated as steady. The coordinate frames are related through

$$\bar{z} = \bar{Z} - c\bar{t}, \qquad \bar{r} = \bar{R}, \tag{2.2}$$

$$\bar{w} = \bar{W} - c, \qquad \bar{u} = \bar{U}, \tag{2.3}$$

where $\bar{u},\bar{w}$ are respectively the radial and the axial velocity components in the moving frame.

The non-dimensional forms of equation of continuity, equations of motion and the boundary conditions, respectively, are [1]

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2.4}$$

$$Re\,\delta^3\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \delta^2\left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \delta^2\frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right) - \delta^2\frac{u}{k}, \tag{2.5}$$

$$Re\,\delta\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \delta^2\frac{\partial^2 w}{\partial z^2}\right) - \frac{w}{k}, \tag{2.6}$$

$$\frac{\partial w}{\partial r} = 0, \qquad u = 0 \qquad \text{for} \qquad r = 0, \tag{2.7a}$$

$$w = -1, \qquad u = -\frac{dh}{dz} \qquad \text{for} \qquad r = h, \tag{2.7b}$$

where the non-dimensional parameters

$$z=\frac{\bar{z}}{\lambda},\bar{Z}=\frac{\bar{Z}}{\lambda},r=\frac{\bar{r}}{d},R=\frac{\bar{R}}{d},t=\frac{c\bar{t}}{\lambda},p=\frac{d^2\bar{p}}{c\lambda\mu},u=\frac{\lambda\bar{u}}{dc},w=\frac{\bar{w}}{c},\varphi=\frac{b}{d},$$

$$W=\frac{\bar{W}}{c},\ \delta=\frac{d}{\lambda},\ Re=\frac{\rho cd}{\mu},h=\frac{\bar{h}}{d}=1+\varphi\sin 2\pi z$$

have been used.

Note that $\bar{p}$ is the pressure, μ is the viscosity, ρ is the density, $\bar{k}$ is the permeability parameter and

δ is the wave number,

Re is the Reynolds number,

$\varphi=\frac{b}{d}<1$, is the amplitude ratio.

3. Rate of Volume Flow

The instantaneous volume flow rate in the fixed frame is given by

$$Q^*=2\pi\int_0^{\bar{h}}\overline{WR}\,d\bar{R}\,, \tag{3.1}$$

where $\bar{h}$ is a function of $\bar{Z}$ and $\bar{t}$.

The rate of volume flow in the moving frame (wave frame) is given by

$$\bar{q}^*=2\pi\int_0^{\bar{h}}\bar{w}\,\bar{r}\,d\bar{r}\,, \tag{3.2}$$

where $\bar{h}$ is a function of $\bar{z}$.

Using equation (3.2), one finds that the two rates of volume flow are related by

$$Q^*=\bar{q}^*+\pi c\bar{h}^2\,. \tag{3.3}$$

The time-mean flow over a period $T=\frac{\lambda}{c}$ at a fixed position $\bar{Z}$ is defined as

$$\bar{Q}^*=\frac{1}{T}\int_0^T Q^*\,d\bar{t}\,, \tag{3.4}$$

which can be written, using (2.1) and (3.3), as

$$\overline{Q}^* = \overline{q}^* + \pi c d^2 \left(1 + \frac{\varphi^2}{2}\right). \tag{3.5}$$

Defining the dimensionless time-mean flows θ and f in the fixed and wave frame respectively as

$$\theta = \frac{\overline{Q}^*}{\pi c d^2} \qquad \text{and} \qquad f = \frac{\overline{q}^*}{\pi c d^2}, \tag{3.6}$$

then making use of (3.6), equation (3.5) can be rewritten as

$$\theta = f + 1 + \frac{\varphi^2}{2}, \tag{3.7}$$

where

$$f = 2\int_0^{h(z)} r\,w\,dr. \tag{3.8}$$

4. Perturbation Solution

Assuming the wave number δ to be small, we obtain the solution for the problem as a power series in terms of δ by expanding the following quantities as

$$u = u_0 + \delta u_1 + O(\delta^2)$$

$$w = w_0 + \delta w_1 + O(\delta^2)$$

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z} + \delta \frac{\partial p_1}{\partial z} + O(\delta^2)$$

$$f = f_0 + \delta f_1 + O(\delta^2). \tag{4.1}$$

Now using the perturbation expansions (4.1) in equations (2.14), (2.16), (2.22), (2.17-21) and (3.8) and collecting terms of like powers of δ, we obtain two sets of coupled linear differential equations with their corresponding boundary conditions in u_0, w_0 and u_1, w_1 for the first two powers of δ as follows

System of Order Zero

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,u_0)+\frac{\partial w_0}{\partial z}=0, \tag{4.2}$$

$$\frac{\partial p_0}{\partial r}=0, \tag{4.3}$$

$$\frac{\partial p_0}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w_0}{\partial r})-\frac{w_0}{k}, \tag{4.4}$$

The corresponding boundary conditions are

$$\frac{\partial w_0}{\partial r}=0, \qquad u_0=0 \qquad \text{for} \qquad r=0, \tag{4.5a}$$

$$w_0=-1, \qquad u_0=-\frac{dh}{dz} \qquad \text{for} \qquad r=h. \tag{4.5b}$$

The solution of equations (4.2-4), subject to the boundary conditions (4.5), is

$$w_0=c_0+c_1\,I_0\left(\frac{r}{\sqrt{k}}\right), \tag{4.6a}$$

and

$$u_0=-\frac{c_0'}{2}r-c_1'\sqrt{k}\,I_1\left(\frac{r}{\sqrt{k}}\right), \tag{4.6b}$$

where

$$c_0=b_0\,f_0+b_1\,,\,c_1=a_0\,f_0+a_1\,,a_0=\left[2\sqrt{k}\,h\,I_1\left(\frac{h}{\sqrt{k}}\right)-h^2\,I_0\left(\frac{h}{\sqrt{k}}\right)\right]^{-1},$$

$$a_1=h^2\,a_0\,,\;b_0=-a_0\,I_0\left(\frac{h}{\sqrt{k}}\right)\,,\;b_1=-2\sqrt{k}\,h\,I_1\left(\frac{h}{\sqrt{k}}\right)a_0\,,$$

$$f_0=k\left(\frac{dp_0}{dz}\right)\left[2\sqrt{k}\,hI_1\left(\frac{h}{\sqrt{k}}\right)\Big/\,I_0\left(\frac{h}{\sqrt{k}}\right)-h^2\right]-2\sqrt{k}\,hI_1\left(\frac{h}{\sqrt{k}}\right)\Big/\,I_0\left(\frac{h}{\sqrt{k}}\right),$$

(4.7)

I_0 and I_1 are modified Bessel polynomials of order zero and one respectively.

Solving (4.7) for $\frac{dp_0}{dz}$ we obtain

$$\frac{dp_0}{dz} = \frac{a_0}{k}\left[f_0 I_0\left(\frac{h}{\sqrt{k}}\right) + 2\sqrt{k}\, h I_1\left(\frac{h}{\sqrt{k}}\right)\right]. \tag{4.8}$$

System of Order One

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_1) + \frac{\partial w_1}{\partial z} = 0, \tag{4.9}$$

$$\frac{\partial p_1}{\partial r} = 0, \tag{4.10}$$

$$Re\left(u_0 \frac{\partial w_0}{\partial r} + w_0 \frac{\partial w_0}{\partial z}\right) = -\frac{\partial p_1}{\partial z} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w_1}{\partial r}) - \frac{w_1}{k}\right), \tag{4.11}$$

with the boundary conditions

$$\frac{\partial w_1}{\partial r} = 0, \qquad u_1 = 0 \qquad \text{for} \qquad r = 0 \tag{4.12a}$$

$$w_1 = 0, \qquad u_1 = 0 \qquad \text{for} \qquad r = h. \tag{4.12b}$$

On substituting the zeroth-order solution (4.6) into equation (4.11) and using (4.10), one finds

$$\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} - \frac{w_1}{k} = \frac{dp_1}{dz} + Re\left[\left\{c_1 I_0\left(\frac{r}{\sqrt{k}}\right) + c_0\right\}\left\{c_1' I_0\left(\frac{r}{\sqrt{k}}\right) + c_0'\right\} - \left\{c_1'\sqrt{k} I_1\left(\frac{r}{\sqrt{k}}\right) + \frac{c_0'}{2} r\right\}\left\{\frac{c_1}{\sqrt{k}} I_0'\left(\frac{r}{\sqrt{k}}\right)\right\}\right], \tag{4.13}$$

The second term of equation (3.13) may be expanded as a power series in r^2. Accordingly, it can be rewritten as

$$\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}-\frac{w_1}{k}=\frac{dp_1}{dz}+\sum_{n=0}^{\infty}B_{2n}\,r^{2n}, \tag{4.14}$$

Equation (4.14) has the solution

$$w_1=k\left(\frac{dp_1}{dz}\right)\left[I_0\left(\frac{r}{\sqrt{k}}\right)\Big/I_0\left(\frac{h}{\sqrt{k}}\right)-1\right]+\sum_{n=1}^{\infty}T_n\,r^{2n}-\left[I_0\left(\frac{r}{\sqrt{k}}\right)\Big/I_0\left(\frac{h}{\sqrt{k}}\right)\right]\sum_{n=1}^{\infty}T_n\,h^{2n} \tag{4.15}$$

The instantaneous volume flow rate f_1 in the moving coordinates is given by

$$f_1=2\int_0^h r\,w_1\,dr=k\left(\frac{dp_1}{dz}\right)\left[2\sqrt{k}\,h\,I_1\left(\frac{h}{\sqrt{k}}\right)\Big/I_0\left(\frac{h}{\sqrt{k}}\right)-h^2\right]+\sum_{n=1}^{\infty}\frac{T_n\,h^{2n+2}}{(n+1)}$$

$$-\left[2\sqrt{k}\,I_1\left(\frac{h}{\sqrt{k}}\right)\Big/I_0\left(\frac{h}{\sqrt{k}}\right)\right]\sum_{n=1}^{\infty}T_n\,h^{2n+1},$$

which implies that

$$\frac{dp_1}{dz}=\frac{a_0 I_0\left(\frac{h}{\sqrt{k}}\right)}{k}\left[f_1-\sum_{n=1}^{\infty}\frac{T_n\,h^{2n+2}}{(n+1)}\right]+\frac{2a_0 I_1\left(\frac{h}{\sqrt{k}}\right)}{\sqrt{k}}\sum_{n=1}^{\infty}T_n\,h^{2n+1}. \tag{4.16}$$

On substituting (4.16) into (4.15), we obtain the alternative form of w_1 as

$$w_1=c_2+b_0\,f+(a_0\,f_1+c_3)I_0\left(\frac{r}{\sqrt{k}}\right)_1+\sum_{n=1}^{\infty}T_n\,r^{2n}, \tag{4.17}$$

where

$$c_2=a_0\,I_0\left(\frac{h}{\sqrt{k}}\right)\sum_{n=1}^{\infty}\frac{T_n\,h^{2n+2}}{(n+1)}-2\sqrt{k}\,a_0\,I_1\left(\frac{h}{\sqrt{k}}\right)\sum_{n=1}^{\infty}T_n\,h^{2n+1},$$

and

$$c_3=-a_0\sum_{n=1}^{\infty}\frac{T_n\,h^{2n+2}}{(n+1)}+\left[\left\{2\sqrt{k}\,h\,a_0\,I_1\left(\frac{h}{\sqrt{k}}\right)-1\right\}\Big/I_0\left(\frac{h}{\sqrt{k}}\right)\right]\sum_{n=1}^{\infty}T_n\,h^{2n}. \tag{4.18}$$

Now, the axial velocity component w and the pressure gradient $\frac{dp}{dz}$ can be expressed, to frist order where $f_0=f-\delta f_1$, as

$$w = c_0 + c_1 I_0\left(\frac{r}{\sqrt{k}}\right) + \delta\left(c_2 + c_3 I_0\left(\frac{r}{\sqrt{k}}\right) + \sum_{n=1}^{\infty} T_n r^{2n}\right), \tag{4.19}$$

$$\frac{dp}{dz} = \frac{a_0 I_0\left(\frac{h}{\sqrt{k}}\right)}{k} f + \frac{2h a_0}{\sqrt{k}} I_1\left(\frac{h}{\sqrt{k}}\right)$$

$$-\delta\left[\frac{a_0}{k} I_0\left(\frac{h}{\sqrt{k}}\right)\sum_{n=1}^{\infty}\frac{T_n h^{2n+2}}{(n+1)} - \frac{2a_0}{\sqrt{k}} I_1\left(\frac{h}{\sqrt{k}}\right)\sum_{n=1}^{\infty} T_n h^{2n+1}\right]. \tag{4.20}$$

The pressure rise per wavelength Δp_λ and friction force F_λ are given by

$$\Delta p_\lambda = \int_0^1 \frac{dp}{dz} dz, \tag{4.21}$$

and

$$F_\lambda = \int_0^1 h^2\left(-\frac{dp}{dz}\right) dz. \tag{4.22}$$

Using (4.20) into (4.21) and (4.22) then evaluating the integrals numerically we obtain both pressure rise and friction force. The results are discussed through Figs.1-4.

5. Results and Conclusion

It is clear that our results calculate the velocity and the pressure gradient without restrictions on the amplitude ratio and the Reynolds number. The only restriction we used that the wave number is less than one.

To discuss the results obtained quantitatively, we evaluate the integrals (4.21) and (4.22) numerically using the MATHEMATICA package.

In Fig.(1), the dimensionless pressure rise (Δp_λ) is graphed versus the dimensionless flow rate (θ) for different values of permeability parameter (k=0.05, 0.1, 1) at wave number δ=0.02 and Reynolds number Re=1, for the case $\varphi = 0.2$. As shown, the pressure rise increases as the permeability decreases. This is because of the resistance caused by the porous medium.

Fig.(2) shows the relation between pressure rise (Δp_λ) and flow rate (θ) at Re=1, δ=0.02 and $\varphi = 0.8$ with the same different values of k. It is clear that the pumping rate in this case $(\varphi = 0.8)$ is greater than the last one $(\varphi = 0.2)$. Again, we note that the pressure rise increases as the permeability decreases.

The relation between friction force and flow rate given by equation (4.23) is plotted in Fig.(3) and Fig.(4) at Re=10, δ=0.02, (k=0.05, 0.1, 0.5, 1), for $\varphi = 0.2$ and $\varphi = 0.8$ respectively. We notice from these Figs. that the friction force increases as the flow rate increases. Also, it is noted that the friction force increases as the permeability decreases.

The dimensionless friction force is plotted versus flow rate in Fig.(3) and Fig.(4) in the two cases $(\varphi = 0.2$ and $\varphi = 0.8)$ at Re=1, δ=0.02 and for various values of k. The graphics reveal that the friction force has the opposite behavior compared to the pressure rise.

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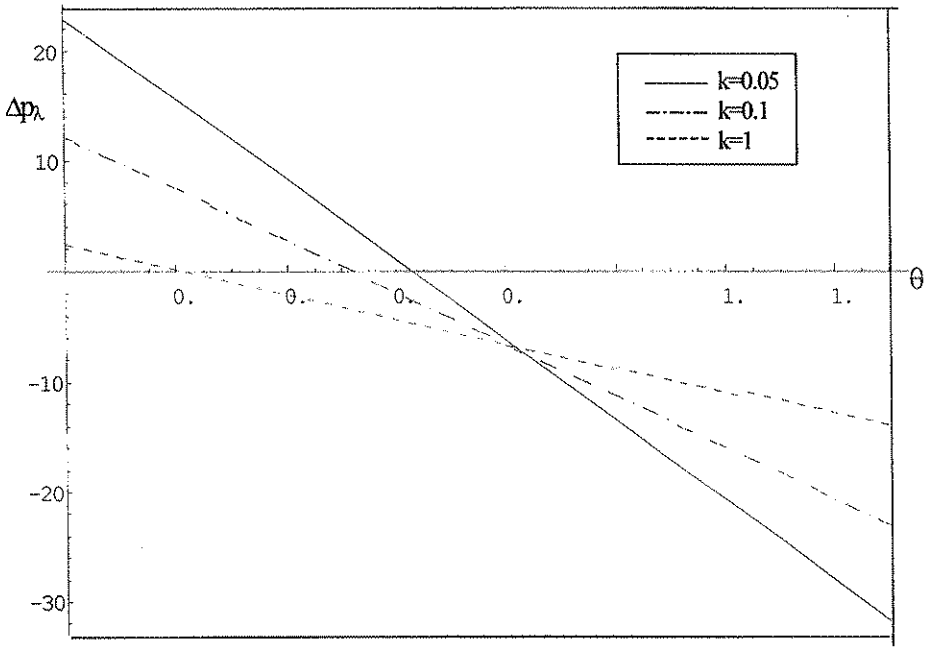

Fig.(1) The pressure rise versus flow rate at Re=1, δ=0.02 and φ=0.2

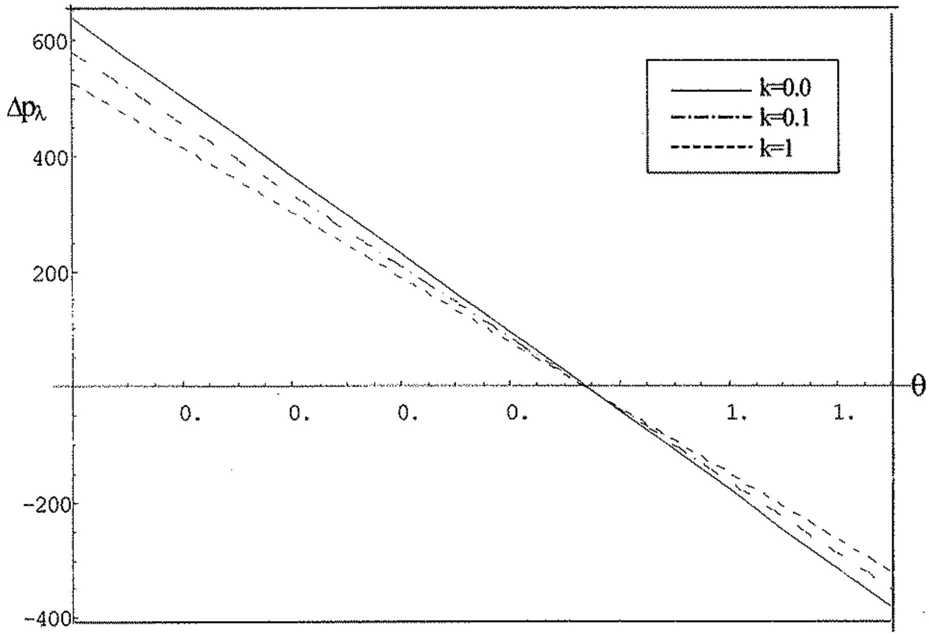

Fig.(2) The pressure rise versus flow rate at Re=1, δ=0.02 and φ=0.8

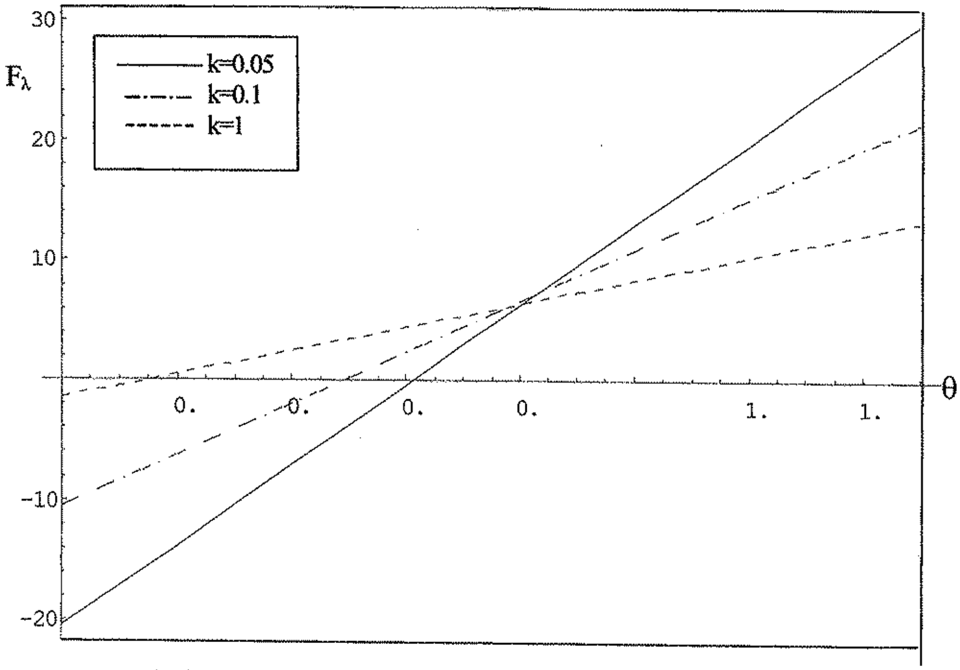

Fig.(3) The friction versus flow rate at Re=1, δ=0.02 and φ=0.2

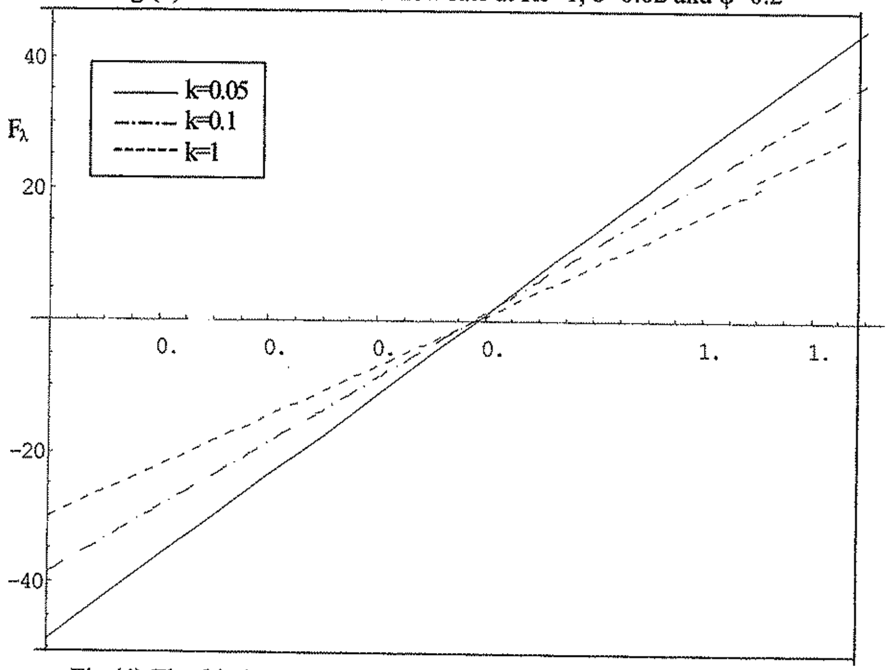

Fig.(4) The friction force versus flow rate at Re=1, δ=0.02 and φ=0.8