# Shadow Spaces of Alexandroff Spaces
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## ABSTRACT

It is well known that a $T_0-$Alexandroff space $\tau$ introduces a partial order $\leq_\tau$ on $X$. This order together with the graph of the poset $(X, \leq_\tau)$ is used in [1], [2], [3] and [4] to give a new formulation for some topological properties on $X$. This proves to be an easier approach.

In this paper, we construct a $T_0-$Alexandroff space $[X]$ from an Alexandroff space $X$. We use the new class with the help of the results of [1], [2], and [3] to derive some properties of $X$ such as:

1. The Alexandroff space $X$ is first countable.
2. $X$ is second countable if and only if $[X]$ is countable.
3. $X$ is hyperconnected if and only if $[X]$ is hyperconnected.
4. X is regular if and only if [X] is regular.
5. An Alexandroff space which is not $T_0$ can not be submaximal.

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# 1.INTRODUCTION:

An Alexandroff space 9 (or minimal neighborhood space ) $X$ is a space in which arbitrary intersection of open sets is open. This is equivalent to saying that every point of X has a minimal neighborhood.

For each $T_0-$Alexandroff space $(X, \tau)$, there corresponds a poset $(X, \leq_\tau)$, in 1-1 and onto way, where each one of them is completely determined by the other. Given a $T_0$-Alexandroff space (X, $\tau$), the partial order $\leq_\tau$ (called Alexandroff specialization order) is defined by :

$$a \leq_\tau b \text{ iff } a \in \overline{\{b\}} .$$

We denote a $T_0-$*Alexandroff space* with its specialization order to be the pair $(X, \tau(\leq))$ where the corresponding poset is $(X, \leq)$.

$T_0-$Alexandroff spaces have been the subject of many research papers recently, see 5. They are used for a topology on the computer screen and for the digital topology.

The importance of studies in [1-4] comes from not only the results we got but from the technique we used. Our method of studies and proofs depends basically on the corresponding poset rather than the topology itself, which proves to be an easier approach. This technique requires the Alexandroff space $X$ to be $T_0$ . The results of [1], [2], [3], and [4] do not include the Alexandroff space $X$ which is not $T_0$, since in this case, the Alexandroff specialization order induced by $X$ is not partially ordered.

In this paper, we will remedy this problem partially by constructing a $T_0$-Alexandroff space $[X]$ from a given Alexandroff space $X$ . We use the results of [1-4] for $[X]$. We will study the common properties between $[X]$ and $X$.

Throughout this paper, the symbol $(X, \tau(\leq))$ denotes a $T_0-$ Alexandroff space where $\leq$ is its (Alexandroff) specialization order. For each element $x \in X$ , $\uparrow x$ or V(x) denotes the minimal neighborhood of $x$ ($\uparrow x = \{y : x \leq y\}$). For a subset $A$ of $X$ the interior (resp. the closure, the derive, the boundary) will be denoted by $A^o$ (resp. , $\overline{A}, A', bd(A)$ ).

## 2.Definitions And Preliminaries:

Recall that an Alexandroff space $X$ is $T_1$ if and only if $X$ is discrete if and only if the specialization order $\leq$ which is induced by $X$ is anti-chain.

**2.1 Definition.** The space $(X, \tau)$ is called a *regular space* if for each closed subset $F \subseteq X$ and each $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subseteq U$.

**2.2 Definitions.** The space $(X, \tau)$ is called.

1. hyperconnected if each open set is dense.

2. submaximal, 10 if each dense subset is open.

**2.3 Theorem.** [1] Let $(X, \tau(\leq))$ be a $T_0$-Alexandroff space, $x \in X$ and $A \subseteq X$, then

1. $\overline{x} = \downarrow x$ $(\downarrow x = \{y : y \leq x\})$.

2. $\overline{A} = \downarrow A = \bigcup\{\downarrow x : x \in A\} = \{x : x \leq y$ for some $y \in A\}$.

**2.4 Theorem.** [1] Let $(X, \tau(\leq))$ be a $T_0$-Alexandroff space. The following are equivalent:

1. $X$ is submaximal.

2. Each element of $X$ is either maximal or minimal.

**2.5 Theorem.** [1] Let $(X, \tau(\leq))$ be a $T_0$-Alexandroff space. If $X$ contains a maximum element $T$, then $X$ is hyperconnected.

**2.6 Theorem.** A $T_0$-Alexandroff space $(X, \tau(\leq))$ is regular if and only if X is discrete.

Proof. ($\Rightarrow$) Suppose that $x \leq y$ in $X$, then $\downarrow y$ is closed set and $x \notin \downarrow y$. If U is open set contains $\downarrow y$ then $\uparrow y \subseteq U$. Hence $x \in U$. Therefore X is not regular.

($\Leftarrow$) This direction is true in any topological space.

## 3. Shadow of Alexandroff Spaces:

Suppose that $(X, \tau)$ is an Alexandroff space (need not be $T_0$), we will construct a $T_0-$Alexandroff space from $X$, 4. Define a relation on $X$ as follows, for each $a, b \in X$, $a \sim b$ if and only if $a, b$ can't be separated, i.e., there is no open neighborhood $U$ which contains one and not the other. This relation is an equivalence relation, since $a \sim a$ and if $a \sim b$ then $b \sim a$. For transitive property, suppose that $a \sim b$ and $b \sim c$ and suppose that $U$ is a neighborhood of $a$ so that $b \in U$ and hence $c \in U$, so $a \sim c$. Denote the minimal neighborhood of an element $a \in X$ by $V(a)$. If $a \sim b$ then $b \in V(a)$ and $a \in V(b)$ and hence we get $V(a) \subseteq V(b)$ and $V(b) \subseteq V(a)$ therefore $a \sim b$ if and only if $V(a) = V(b)$.

**3.1 Definition**. 4 Let $[X] := \{[a] : a \in X\}$ be the set of all equivalence classes of an Alexandroff space $X$ with respect to the equivalence relation $\sim$. Define the notation $\leq$ on $[X]$ by

$[a] \leq [b]$ if and only if $b \in V(a)$ (equivalently, $V(b) \subseteq V(a)$ ).

It is not difficult to prove that this definition is well-defined.

**3.2 Proposition**. $[a] \leq [b]$ if and only if $a \in \overline{b}$

Proof. ($\Rightarrow$) If $a \notin \overline{b}$, then $a \in \overline{\{b\}}^c$, V (a) $\subseteq \overline{\{b\}}^c$, so b $\notin$ V (a) therefore V (b) V(a) and hence [a] $\hbar$ [b].

(i) ($\Leftarrow$) If [a] $\hbar$ [b] so V (b) V (a) which implies that b $\notin$ V(a) or b $\in V(a)^c$ and we get that $\overline{b} \subseteq V(a)^c$, therefore $a \notin \overline{b}$.

**3.3 Lemma**. The set $[X]$ together with the notation $\leq$ in the Definition 3.1 form a partially order set.

*Proof.* For $x, y, z \in X$

(ii) $V(x) \subseteq V(x)$ so $[x] \leq [x]$.

(iii) If $[x] \leq [y]$ and $[y] \leq [x]$ then $V(x) \subseteq V(y)$ and $V(y) \subseteq V(x)$ so $V(x) = V(y)$ and so that $x \sim y$ therefore $[x] = [y]$.

(iv) If $[x] \leq [y]$ and $[y] \leq [z]$ then $V(y) \subseteq V(x)$ and $V(z) \subseteq V(y)$ therefore $V(z) \subseteq V(x)$ and hence $[x] \leq [z]$.

If $A \subseteq X$, we will set $[A] := \{[a] : a \in A\}$. Now $([X], \leq)$ is a poset and so a $T_o$−Alexandroff topology $\tau(\leq)$ with respect to the order $\leq$ is induced on $[X]$. y convention, if $G \subseteq [X]$, we will write $G = [A]$ where $A = \{c \in X : [c] \in G\}$ as main representation of $G$. Note that this representation is not unique. To see this, suppose that $G = [A]$ and $a, b \in A$ are two different points such that $V(a) = V(b)$, then $G = [A']$ where $A' = A - \{b\}$.

If $[A_1] = [A_2]$, are two different representations of a subset of $[X]$, and if $a \in A_1$ and $a \notin A_2$, then there exists $b \in A_2$ such that $[a] = [b]$ (or $V(a) = V(b)$).

**3.4 Definition.** Let $(X, \tau)$ be an Alexandroff space and let $([X], \leq)$ be its corresponding poset. The $T_o$−Alexandroff space $([X], \tau(\leq))$ is called shadow space ( or $T_o$-quotient ) of X.

**3.5 Theorem**. Let $(X, \tau)$ be an Alexandroff space and let $([X], \tau(\leq))$ be its corresponding shadow space. The function f : $(X, \tau) \to ([X], \tau(\leq))$ which takes $x \to [x]$ is a continuous open map.

*Proof.* $\uparrow[a] = \{[c] : [a] \leq [c]\} = \{[c] : c \in V(a)\}$, so $[c] \in \uparrow[a] \Leftrightarrow c \in V$ $(a)$. Therefore

$$f^{-1}(\uparrow[a]) = \{c \in X : [c] \in \uparrow[a]\} = \{c \in X : c \in V(a)\} = V(a) \in \tau$$

Moreover $f(V(a)) = \{[c] : c \in V(a)\} = \uparrow[a] \in \tau(\leq)$. This proves that $f$ is

open and continuous.

The following theorem is important for our main results, so we give the proof of this theorem.

**3.6 Theorem**.4 Let $(X, \tau)$ be an Alexandroff space and $([X], \leq)$ be its corresponding poset with $T_o$−Alexandroff topology $\tau(\leq)$. There is a $1$−$1$ correspondence between $\tau$ and $\tau(\leq)$.

*Proof.* Define $F : \tau \to \tau(\leq)$ as follows, if $U \in \tau$ and $U = \bigcup_\alpha V(a_\alpha)$, then

$$F(U) = \bigcup_\alpha \uparrow[a_\alpha]$$

<u>*F* is well-defined:</u> Suppose that $u = \bigcup_\alpha V(a_\alpha) = \bigcup_\beta V(b_\beta)$. We want to

prove that $\bigcup_\alpha \uparrow[a_\alpha] = \bigcup_\beta \uparrow[b_\beta]$. It suffices to prove that $\bigcup_\alpha \uparrow[a_\alpha] \subseteq$

$\bigcup_\beta \uparrow[b_\beta]$. Let $[c] \in \bigcup_\alpha \uparrow[a_\alpha]$, so there exists $\alpha_o$ such that $[c] \in \uparrow[a_{\alpha_o}] \Rightarrow$

$[a_{\alpha_o}] \leq [c]$, thus we have the following

$$c \in V(c) \subseteq V(a_{\alpha_o}) \subseteq \bigcup_\alpha V(a_\alpha) = \bigcup_\beta V(b_\beta)$$

Hence, there exists $\beta_o$ such that $c \in V(b_{\beta_o})$, so $[b_{\beta_o}] \leq [c]$ and we get

$$[c] \in \uparrow[b_{\beta_o}] \subseteq \bigcup_\beta \uparrow[b_\beta]$$

Therefore $\bigcup_\alpha \uparrow[a_\alpha] = \bigcup_\beta \uparrow[b_\beta]$ and $F$ is well-defined.

<u>*F* is onto:</u> Let $O \in \tau(\leq)$, so $O = \bigcup_\alpha \uparrow[a_\alpha]$ and we get $\bigcup_\alpha V(a_\alpha) \in \tau$,

moreover

$$F(\bigcup_\alpha V(a_\alpha)) = \bigcup_\alpha \uparrow[a_\alpha] = O.$$

<u>F is $1$ − $1$:</u> Suppose that $F(U_1) = F(U_2)$ for some $U_1, U_2 \in \tau$. It suffices to prove that if $x \in U_1$ then $x \in U_2$. Suppose that $x \in U_1$ and $U_2 = \bigcup_\alpha V(a_\alpha)$. $U_1$ is open so $V(x) \subseteq U_1$ and hence

$$\uparrow[x] = F(V(x)) \subseteq F(U_1) = F(U_2) = \bigcup_{\alpha} \uparrow [a_\alpha]$$

which implies that $[x] \in \bigcup_{\alpha} \uparrow[a_\alpha]$, so there exists $\alpha_o$ such that $[x] \in \uparrow[a_{\alpha_o}]$

which is equivalent to that $[a_{\alpha_o}] \leq [x]$, therefore $V(x) \subseteq V(a_{\alpha_o}) \subseteq U_2$ and

hence $x \in U_2$.

## 4. Main Results:

In this part, we use Theorem 3.6 to derive some topological properties of Alexandroff spaces by the help of the results of $T_o$−Alexandroff spaces in [1-3]. In the following theorems, we will study some of related properties between Alexandroff and $T_o$-Alexandroff spaces.

**4.1 Theorem**. For the Alexandroff space $X$, we have

1. $X$ is first countable.
2. $X$ is second countable if and only if $[X]$ is countable.
3. if $x \in X$ then $\overline{x} = \{c : V(x) \subseteq V(c)\}$.
4. if $A \subseteq X$ then $\overline{A} = \{c : V(x) \subseteq V(c) \text{ for some } x \in A\}$.

*Proof.* Parts (1) and (2) come directly from Theorem *3.6*

(3) From Definition 3.1 and Proposition 3.2, $V(x) \subseteq V(c)$ if and only if

$[c] \leq [x]$ if and only if $c \in \overline{x}$.

(4) Comes from the fact that in Alexandroff space $\overline{A} = \bigcup_{x \in A} \overline{x}$.

**4.2 Corollary.** Let $X$ be an Alexandroff space. Then

1. $a \in \overline{x}$ if and only if $[a] \in \overline{[x]}$.
2. for any $A \subseteq X$, $a \in \overline{A}$ if and only if $[a] \in \overline{[A]}$.
3. for any $A \subseteq X$, $A$ is dense in $X$ if and only if $[A]$ is dense in $[X]$.

*Proof.* (1) Direct result of Theorem 4.1 part 3.

(2) ($\Rightarrow$) If $a \in \overline{A}$, then by Theorem 4.1 part 4, there exists $x \in A$ such that

$V(x) \subseteq V(a)$. This implies that $[a] \leq [x]$. Hence $[a] \in \overline{[A]}$.

($\Leftarrow$) The difficulty of this direction comes from the fact that $[A]$ has

different representations in $[X]$. Let $[c] \in \overline{[A]}$. By Theorem 2.3 part 2, there

exists $[a] \in [A]$ such that $[c] \leq [a]$. Note that if $[x] \in [A]$ ( $x$ need not be in

$A$ ) then there exists $y \in A$ such that $[x] = [y]$. So we can choose $a' \in A$

such that $[a'] = [a]$. Hence, $[c] \leq [a']$ and so by Proposition 3.2, $c \in \overline{a'} \subseteq \overline{A}$.

(3) easy.

**4.3 Theorem.** If $X$ is an Alexandroff space, then
1. $X$ is hyperconnected if and only if $[X]$ is hyperconnected,
2. $X$ is regular if and only if $[X]$ is regular, in this case, $[X]$ is discrete.

*Proof.* (1) It is true, using Corollary 4.2 part 3, that each open set $A$ in $X$ is dense if and only if the corresponding open set $[A]$ in $[X]$ is dense.

(2) ($\Rightarrow$) Suppose by contrary that $[a] \leq [b]$ in $[X]$. Then $[b] \notin \downarrow [a] = \overline{[a]}$. By Corollary4.2 part 1, $b \notin \overline{a}$. So there exist two disjoint open sets $U_1, U_2$ in $X$ such that $b \in U_1$ and $\overline{a} \subseteq U_2$. Since $[a] \leq [b]$, we get $V(b) \subseteq V(a) \subseteq U_2$. But $V(b) \subseteq U_1$ and this contradicts that $U_1$ and $U_2$ are disjoint.

($\Leftarrow$) If $[X]$ is regular then by Theorem 2.6 $[X]$ is discrete and hence the specialization order is anti-chain. So for each $x \in X$, $V(x)$ is clopen. Hence $X$ is regular.

**4.4 Corollary.** Let $X$ be an Alexandroff space. If there exists $c \in X$ such that $c \in V(x)$ for each $x \in X$ then $X$ is hyperconnected.

*Proof.* $V(c) \subseteq V(x)$ $\forall x \in X$, so $[x] \leq [c]$ in $[X]$. Hence $[c]$ is a maximum element in $[X]$. By Theorem 2.5 $[X]$ is hyperconnected and by Theorem 4.3 part 1 $X$ is hyperconnected.

**4.5 Remark.** Although, one is might think that "if $X$ is submaximal then $[X]$ is submaximal", since if $[A]$ is dense in its corresponding shadow space $[X]$, then $A$ is dense and hence open in $X$. Therefore $[A]$ is open. This seems ok. In fact, this sentence is useless as the following proposition illustrates.

**4.6 Proposition.** An Alexandroff space which is not $T_0$ can not be submaximal.

*Proof.* Let $X$ be an Alexandroff space such that $a \neq b$ in $X$ and $V(a) = V(b)$. Then $X - \{b\}$ is dense in $X$. Since $b \in V(a)$, we get that $V(a) \quad X - \{b\}$. This implies that $X - \{b\}$ is not open. Hence $X$ is not submaximal.

The following two examples give Alexandroff spaces, one is finite and the other one is infinite, where their corresponding shadow spaces are finite submaximal.

**4.7 Example**. Let $X = \{a, b, c, d\}$ be a set with Alexandroff topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. $X$ is not $T_o$ and $[a] = \{a\}$, $[b] = \{b\}$, and $[c] = \{c, d\}$. Moreover $V(a) = \{a\}$, $V(b) = \{b\}$, $V(c) = X$, and $V(d) = X$, so the order on $[X] = \{[a], [b], [c]\}$ is $[c] \leq [a]$, $[c] \leq [b]$, $[c] = [d]$, and $[a]$, $[b]$ are incomparable. The graph of the poset $[X]$ is shown in the following figure



$[a]$          $[b]$

$[c] = [d]$
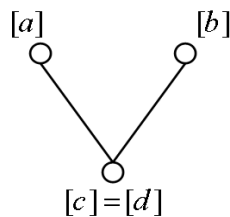
We can use this figure to see that $\overline{a} = \{a, c, d\}$. Clearly $[X]$ is submaximal.

**4.8 Example**. Let $X = \mathbb{N}$ be the set of natural numbers with Alexandroff topology of minimal neighborhood base $\doteq = \{\mathbb{N}, O\}$, where $O$ is the set of odd natural numbers. So $\tau = \{\phi, X, O\}$. $X$ is not $T_o$. For any odd number $a$, $V(a) = O$, and for any even number $b$, $V(b) = \mathbb{N}$. So $[X] = \{[a], [b]\}$, where $a$ is any odd number and $,b$ is any even number in $\mathbb{N}$. Moreover $V(a) \subseteq V(b)$, we get $[b] \leq [a]$. The figure of the $T_0 - $Alexandroff space $([X], \tau(\leq))$ is



$[a]$

$[b]$

Note that there is a 1-1 correspondence between $\tau$ and $\tau(\leq)$, where $\tau(\leq)$ is the Sierpinski topology on $[X]$, so it is submaximal. Since $\{2, 3, 4, 5, ...\}$ is dense in $X$ which is not open, $X$ is not submaximal.

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