On non_archimedean $\Lambda(\alpha)-$ Gelfand-Philips Spaces

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الملخص

في الفراغات المحددة المحلية الغير اقليديـة ، المجموعـات والمـؤثرات $\Lambda(\alpha)-$ limited قدمت ودرست ، برهنا أن حاصل ضرب أي عدد منتهي من الفراغـات $\Lambda(\alpha)-GP$ تكـون $\Lambda(\alpha)-GP$ وبرهنا أيضاً تحت بعض الشروط أنه إذا كان $T_1,T_n,....,T_n$ أي عدد منتهي من المؤثرات $\Lambda(\alpha)-$limited من E_i إلى F_i تحت بعض الشروط فإن $\prod_{i=1}^{n} T_i$ تكون مؤثر $\Lambda(\alpha)-$limited .

ABSTRACT

For a non_archimedean locally convex spaces, the $\Lambda(\alpha)-$limited sets and operators are introduced and studied. We show that the finite product of $\Lambda(\alpha)-GP$ spaces is $\Lambda(\alpha)-GP$ space. We also show, under some conditions, that if $T_1,T_n,....,T_n$ are any finite numbers of $\Lambda(\alpha)-$limited operators from E_i into F_i, then the operator $\prod_{i=1}^{n} T_i$ is $\Lambda(\alpha)-$limited operator.

Key words: non_archimedean locally convex spaces, compactoid, limited sets, Kolmograve diameter.

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INTRODUCTION:

In many branches of mathematics and its applications the valued fields of the real numbers R and the complex numbers C play a fundamental role. For quite some time one has been discussing the consequences of replacing in those theories R or C by the more general object of a non-archimedean valued field $(K,|\ |)$.

In this paper we introduce $\Lambda(\alpha)-$compactoid sets and operators in non-archimedean locally convex spaces. We use the Kolmograve diameters to show that the finite product of any $\Lambda(\alpha)-$compactoid sets is $\Lambda(\alpha)-$compactoid and to obtain for $\Lambda(\alpha)-$compactoid operators results resembling previously known properties of compact operators.

In the classical case of spaces over the real or complex field, analogous problems have been studied by several authors (see, for example, [3], [4], [7], [8]).

Preliminaries:

Let K be a field. A non-archimedean valuation on K is a function $|.|: K \to [0,\infty)$ such that for all $\alpha,\beta \in K$ it satisfies:

$|\alpha| = 0$ if and only if $\alpha = 0$; $|\alpha\beta| = |\alpha|\,|\beta|$; and $|\alpha+\beta| \le \max\{|\alpha|,|\beta|\}$.

Let E be a linear space over the field K. A non-archimedean seminorm on E is a seminorm which verifies the strong triangle inequality: $\|x+y\| \le \max\{\|x\|,\|y\|\}$ for all $x,y \in E$.

Throughout this paper K is a non- archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|.|$, $|K|$ is the set of all scalars $|\lambda|$, $\lambda \in K$. Also $E,F,.....$ are Hausdorff locally convex spaces over K. We will denote by $L(E,F)$ the vector space of all continuous linear operators from E into F and by $cs(E)$ the collection of all continuous non-archimedean seminorm on E. For $p \in cs(E)$ and $r > 0$, $B_p(0,r)$ will be the set $\{x \in E : p(x) \le r\}$. By [5] the collection of all neighborhoods $B_p(0,1)$ ($p \in cs(E)$) is a local base of zero in E.

By [1], a subset B of E is called compactoid if for every zero-neighbourhood U in E there exists a finite set $S \subset E$ such that $B \subset co(S)+U$, where $co(S)$ is the absolutely convex hull of S.

Let us denote by the c_0 the space of all sequences in K converging to zero and by $l_\infty(K)$ the space of all bounded sequences in K.

$\Lambda(\alpha)-$compactoid sets

Let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be an increasing sequence of non-negative real numbers tending to infinity and satisfying

$\alpha_{2n} \le c\alpha_n$ for some $c > 0$ and all $n \in N$. (1)

For $R \in |K|/\{0\}$, $R \ge 1$ and a sequence $\xi = (\xi_n)_{n=1}^{\infty}$ in K, we define $p_R(\xi) = \sup_n R^{\alpha_n}|\xi_n|$. The non-archimedean power series space $\Lambda(\alpha)$ is the space of all sequences ξ in K for which $p_R(\xi) < \infty$ for all $R \in |K|/\{0\}$, $R \ge 1$. On $\Lambda(\alpha)$ we consider the locally convex topology generated by the family $\{p_R : R \in |K|/\{0\}, R \ge 1\}$ of non-archimedean seminorms (Compare [2], [6]). Under this topology $\Lambda(\alpha)$ is a complete Hausdroff locally convex space over K, and condition (1) is equivalent to $\Lambda(\alpha) \times \Lambda(\alpha) \cong \Lambda(\alpha)$ and called stability (see [6]).

In case $\alpha = (1, 1, ..., 1)$, we have $\Lambda(\alpha) = l_\infty(K)$ and in case $\alpha = (\log(n+1))$ we obtain the space of all rapidly decreasing sequences [8]. By $[x]$ we mean the integer part of the real number x such that $[x] = \eta$ if $x = \eta + \mu,\ 0 \le \mu < 1$.

Proposition 2.1.

If $\xi = (\xi_0, \xi_1, \xi_2, ...) \in \Lambda(\alpha)$, then $(\xi_{[\frac{n}{2}]})_{n=0}^{\infty} = (\xi_0, \xi_0, \xi_1, \xi_1 ......) \in \Lambda(\alpha)$.

Proof. Let $\xi \in \Lambda(\alpha)$ .Then for $R \in |K|/\{0\}$, $R \ge 1$, and $n > 1$ we have

$$R^{\alpha_n}\xi_{[\frac{n}{2}]} \le R^{\alpha_{2[\frac{n}{2}]+1}}\xi_{[\frac{n}{2}]} \le R^{\alpha_{2[\frac{n}{2}]+2[\frac{n}{2}]}}\xi_{[\frac{n}{2}]} \le R^{\alpha_{4[\frac{n}{2}]}}\xi_{[\frac{n}{2}]} \le R^{c^2\alpha_{[\frac{n}{2}]}}\xi_{[\frac{n}{2}]} = R^{c^2}R^{\alpha_{[\frac{n}{2}]}}\xi_{[\frac{n}{2}]} < \infty \quad ,$$

And so $(\xi_{[\frac{n}{2}]})_{n=0}^{\infty} \in \Lambda(\alpha)$.

For a bounded subset B of a locally convex space E over K, a $p \in cs(E)$ and a non- negative integer n, the nth Kolmograve diameter $\delta_{n,p}(B)$ of B with respect to p is the infimum of all $|\mu|$, $\mu \in K$, for which there exists a subspace F of E with $\dim(F) \le n$, such that $A \subset F + \mu B_p(0,1)$ (see [6]). These nth Kolmograve diameters satisfy the following properties:

Proposition 2.2.

(Compare [3], p16, [7], p208] , [8]):

(i) $\delta_{0,p}(B) \geq \delta_{1,p}(B) \geq \delta_{2,p}(B) \geq .... \geq 0$ for all $p \in cs(E)$.

(ii) If $B_1 \subseteq B$ and $p \leq q$, then $\delta_{n,p}(B_1) \leq \delta_{n,q}(B)$.

(iii) If $T \in L(E,F)$, then for all $p \in cs(F)$ there exists $q \in cs(E)$ such that $\delta_{n,p}(T(B)) \leq \delta_{n,q}(B)$.

Definition 2.3.

(Compare [6]) A subset B of E is called $\Lambda(\alpha)-$ compactoid if for all $p \in cs(E)$ there exists $\xi = (\xi_n)_{n=0}^{\infty} \in \Lambda(\alpha)$ such that $\delta_{n,p}(B) \leq |\xi_n|$ for all n (or equivalently $\sup_n R^{\alpha_n}\delta_{n,p}(B) < \infty$ for all $R \in |K|/\{0\}, R \geq 1$).

In case, $\Lambda(\alpha) = c_0$, the concept of $\Lambda(\alpha)-$ compactoid set coincide with the concept of a compatoid set (see [6]).

Proposition 2.4

Let $E = \prod_{i=1}^{n} E_i$, where each E_i is a locally convex space whose topology generated by the family $cs(E_i)$ of non-archimedean seminorms, and let B_i be any bounded subset of E_i , then for all $p \in cs(E)$ there exist $p_i \in cs(E_i)$, $i = 1,2,......,n$, such that

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \leq \inf imum\{\max(\delta_{k_1,p_1}(B_1),....,\delta_{k_n,p_n}(B_n))\}$$

where the infimum is taken over all choices $k_1 + k_2 + ... + k_n \leq s$.

Proof : Suppose $p \in cs(E)$, then the neighborhood $B_p(0,1)$ of zero in E can be taken in the form $\prod_{i=1}^{n} U_i$, where U_i is a neighborhood of zero in E_i . Since the collection $\{B_{p_i}(0,1) : p_i \in cs(E_i)\}$ is a local base of zero in E_i , then for each U_i, there exists $p_i \in cs(E_i)$ such that $B_{p_i}(0,1) \subset U_i$, so $\prod_{i=1}^{n} B_{p_i}(0,1) \subset \prod_{i=1}^{n} U_i = B_p(0,1)$. Now, according to the definitions of $\delta_{k_i,p_i}(B_i)$, $i = 1,2,......,n$, we have for $\varepsilon > 0$, there exist subspaces $F_1,.........,F_n$ of $E_1,.......,E_n$, respectively, with $\dim(F_1) \leq k_1,.......,\dim(F_n) \leq k_n$, and $\mu_1,......,\mu_n \in K$ such that $|\mu_i| \leq \delta_{k_i,p_i}(B_i) + \varepsilon$ and $B_i \subset \mu_i B_{p_i}(0,1) + F_i$. This implies that

$$\prod_{i=1}^{k} B_i \subset \prod_{i=1}^{n} \mu_i B_{p_i}(0,1) + \prod_{i=1}^{n} F_i \subset \mu B_p(0,1) + \prod_{i=1}^{n} F_i$$

where $|\mu| = \max(|\mu_1|,....,|\mu_n|)$. Since $\dim(\prod_{i=1}^{n} F_i) \le k_1 + k_2 + ... + k_n \le s$, then

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le |\mu| = \max(|\mu_1|,....,|\mu_n|) \le (\max(\delta_{k_1,p_1}(B_1),........,\delta_{k_n,p_n}(B_n))) + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le \max(\delta_{k_1,p_1}(B_1),........,\delta_{k_n,p_n}(B_n)),$$

and since this estimation is true for any choice of $k_1 + k_2 + ... + k_n \le s$, then

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le \min\{\max(\delta_{k_1,p_1}(B_1),........,\delta_{k_n,p_n}(B_n))\}$$

Corollary 2.5

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le \max(\delta_{\left[\frac{s}{2^r}\right],p_1}(B_1),..........,\delta_{\left[\frac{s}{2^r}\right],p_n}(B_n))$$

where r is the smallest integer such that $n < 2^r$.

Proof: Since $\sum_{i=1}^{n} \frac{s}{2^r} \le s$, it follows form proposition (2.4) that

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le \min\{\max(\delta_{k_1,p_1}(B_1),....,\delta_{k_n,p_n}(B_n))\}$$

$$\le \max(\delta_{\left[\frac{s}{2^r}\right],p_1}(B_1),.....,\delta_{\left[\frac{s}{2^r}\right],p_n}(B_n)).$$

Theorem 2.6.

The product of any finite number of $\Lambda(\alpha)-$compactoid sets is $\Lambda(\alpha)-$compactoid.

Proof. Suppose $\prod_{i=1}^{n} B_i \subset \prod_{i=1}^{n} E_i$, where each B_i is a $\Lambda(\alpha)-$compactoid in E_i. Clearly $(\delta_{s,p_i}(B_i))_{s=0}^{\infty} \in \Lambda(\alpha)$ for all $p_i \in cs(E_i)$, it follows from proposition (2.1) that $(\delta_{\left[\frac{s}{2^r}\right],p_i}(B_i))_{s=0}^{\infty} \in \Lambda(\alpha)$ for all $p_i \in cs(E_i)$, where r is the smallest integer such that $n < 2^r$. Now from corollary (2.5), for all $p \in cs(E)$ there exist $p_i \in cs(E_i)$, $i = 1, 2, \ldots, n$, such that

$$\delta_{s,p}(\prod_{i=1}^{n} B_i) \le \max(\delta_{\left[\frac{s}{2^r}\right],p_1}(B_1), \ldots, \delta_{\left[\frac{s}{2^r}\right],p_n}(B_n)) \le \delta_{\left[\frac{s}{2^r}\right],p_1}(B_1) + \ldots + \delta_{\left[\frac{s}{2^r}\right],p_n}(B_n) \quad \forall s = 0, 1, \ldots$$

Thus, $(\delta_{s,p}(\prod_{i=1}^{n} B_i))_{s=0}^{\infty} \in \Lambda(\alpha)$ for all $p \in cs(E)$, which prove that $\prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)-$compactoid in E.

Theorem 2.7

The continuous linear image of any $\Lambda(\alpha)-$compactoid set is $\Lambda(\alpha)-$compactiod.

Proof. Let $T \in L(E,F)$ and let B be a subset of E which is $\Lambda(\alpha)-$compactoid, then $(\delta_{n,p}(B))_{n=0}^{\infty} \in \Lambda(\alpha)$ for all $p \in cs(E)$. Now, by property (iii) of proposition (2.2), for each $q \in cs(F)$ there exists $p \in cs(E)$ such that $\delta_{n,q}(T(B)) \le \delta_{n,p}(B)$ for all $n \in N$, and so $T(B)$ is $\Lambda(\alpha)-$compactoid in F.

Corollary 2.8

If $B = \prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)-$compactoid subset of a locally convex space $E = \prod_{i=1}^{n} E_i$, then B_i is $\Lambda(\alpha)-$compactoid.

Proof: It follows from the fact that, the projection operator P_i from $\prod_{i=1}^{n} E_i$ into E_i is continuous and $P_i(\prod_{i=1}^{n} B_i) = B_i$.

3 $\Lambda(\alpha)-$Limited sets and $\Lambda(\alpha)-$GP-spaces.

Defintion 3.1.

(Compare [6]) An operator $T \in L(E,F)$ between two locally convex spaces E, F over K is called $\Lambda(\alpha)-$compactoid if there exists a neighborhood V of zero in E such that $T(V)$ is $\Lambda(\alpha)-$ compactoid in F.

In case, $\Lambda(\alpha)=c_0$ the concept of $\Lambda(\alpha)-$ compactoid operator coincide with the concept of a compact operator (see [6]).

Definition 3.2

Compare [1]

i) A bounded subset B of E is called $\Lambda(\alpha)-$limited in E if and only if for each continuous linear map T from E to c_0, T(B) is $\Lambda(\alpha)-$compactoid in c_0.

ii) An operator $T\in L(E,F)$ is called $\Lambda(\alpha)-$limited if there exists a zero-neighborhood U in E such that $T(U)$ is $\Lambda(\alpha)-$limited in F

Proposition 3.3.

i) Every $\Lambda(\alpha)-$compactoid subset of E is $\Lambda(\alpha)-$limited in E.

ii) If B is $\Lambda(\alpha)-$limited in E and $T\in L(E,F)$, then T(B) is $\Lambda(\alpha)-$limited in F where $L(E,F)$ denotes the vector space of all continuous linear maps from E to F).

iii) If B is $\Lambda(\alpha)-$limited in E and $D\subset B$, then D is $\Lambda(\alpha)-$limited in E.

iv) Let M be a subspace of E and $B\subset M$. If B is $\Lambda(\alpha)-$limited in M then B is $\Lambda(\alpha)-$limited in E .

Proof:

i) Let B be any $\Lambda(\alpha)-$compactoid subset of E and let $T\in L(E,c_0)$. It follows from property (iii) of proposition (2.2) that for all $p\in cs(F)$ there exists $q\in cs(E)$ such that $\delta_{n,p}(T(B))\leq\delta_{n,q}(B)$ and so T(B) is $\Lambda(\alpha)-$compactoid in c_0.Therefore B is $\Lambda(\alpha)-$limited in E.

ii) Suppose B is $\Lambda(\alpha)-$limited in E and $T\in L(E,F)$. Let $G\in L(F,c_0)$, then $G\circ T\in L(E,c_0)$, so $G(T(B))$ is $\Lambda(\alpha)-$compactoid in c_0. Therefore $T(B)$ is $\Lambda(\alpha)-$limited in E.

iii) Let $D\subset B$ and $T\in L(E,F)$, then $T(D)\subset T(B)$, and hence, by property (ii) of proposition (2.2), $\delta_{n,p}(T(D))\leq\delta_{n,p}(B)$ for all $p\in cs(F)$, and so $T(D)$ is $\Lambda(\alpha)-$compactoid in c_0. And this complete the proof.

iv) Let M be a subspace of E and let $B\subset M$ be $\Lambda(\alpha)-$limited in M. If $T\in L(E,c_0)$, then the restriction operator $T|M\in L(M,c_0)$ is continuous , and so $T|M(B)=T(B)$ is $\Lambda(\alpha)-$compactoid in c_0, Thus B is $\Lambda(\alpha)-$limited in E.

Definition 3.4

Compare [1]

A locally convex space E is called $\Lambda(\alpha)-$Gelfand-Philips space ($\Lambda(\alpha)-$GP-space in short) if every $\Lambda(\alpha)-$limited set in E is $\Lambda(\alpha)-$compactoid.

Theorem 3.5

If $\prod_{i=1}^{n} B_i \subset \prod_{i=1}^{n} E_i$ is $\Lambda(\alpha)-$limited, then B_i is $\Lambda(\alpha)-$limited.

Proof: Let $T: E_i \to c_0$ be any continuous linear operator, then $P_i: \prod E_i \to E_i$ is continuous, then $T \circ P_i: \prod E_i \to c_0$ is continuous, so $T \circ P_i(\prod B_i) = T(B_i)$ is $\Lambda(\alpha)-$compactoid, hence B_i is $\Lambda(\alpha)-$limited.

Proposition 3.6

i) A subspace of $\Lambda(\alpha)-$GP-space is $\Lambda(\alpha)-$GP-space.

ii) The product of a any finite numbers of $\Lambda(\alpha)-$GP-spaces is $\Lambda(\alpha)-$GP-space.

Proof :

i) Let M be a subspace of $\Lambda(\alpha)-$GP-space E and let B be any $\Lambda(\alpha)-$limited M, then ,by (iv) of proposition (3.3) , B is $\Lambda(\alpha)-$limited in E. Since E is $\Lambda(\alpha)-$GP-space, then B is $\Lambda(\alpha)-$compactoid in E, and hence in M, and therefore M is $\Lambda(\alpha)-$GP-space.

ii) Let $E_1, E_2, \ldots\ldots, E_n$ be a finite numbers of $\Lambda(\alpha)-$GP-spaces and let B=$\prod_{i=1}^{n} B_i \subset \prod_{i=1}^{n} E_i$ be $\Lambda(\alpha)-$limited set in $\prod_{i=1}^{n} E_i$. It follows from theorem (3.5) that B_i is $\Lambda(\alpha)-$limited in E_i. Since each E_i is $\Lambda(\alpha)-$GP-space, then B_i is $\Lambda(\alpha)-$compactoid in E_i, Now by theorem (2.6) $\prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)-$compactoid in E. So $\prod_{i=1}^{n} E_i$ is $\Lambda(\alpha)-$GP-space.

Definition 3.7

Let $T_1, T_n, \ldots., T_n$ be any finite numbers of continuous linear operators from E_i into F_i, then The operator $\prod_{i=1}^{n} T_i : \prod_{i=1}^{n} E_i \to \prod_{i=1}^{n} F_i$ is defined by

$$\prod_{i=1}^{n} T_i(x_1, x_2, ..., x_n) = (T_1x_1, T_2x_2, ..., T_nx_n) .$$

Proposition 3.8

Let $T_1, T_n, ..., T_n$ be any finite numbers of $\Lambda(\alpha)-$limited operators from E_i into F_i, where $F_1, F_n, ..., F_n$ are $\Lambda(\alpha)-$GP spaces, then the operator $\prod_{i=1}^{n} T_i$ is $\Lambda(\alpha)-$limited operator.

Proof : Suppose $T_1, T_n, ..., T_n$ are $\Lambda(\alpha)-$limited operators from E_i into F_i, then there exist neighborhoods $V_1, V_n, ..., V_n$ of zero in $E_1, E_n, ..., E_n$ such that $T_1(V_1), T_2(V_2), ..., T_n(V_n)$ are $\Lambda(\alpha)-$limited sets in $F_1, F_2, ..., F_n$. Since $F_1, F_2, ..., F_n$ are $\Lambda(\alpha)-$GP spaces, then $T_1(V_1), T_2(V_2), ..., T_n(V_n)$ are $\Lambda(\alpha)-$compactoid sets, and so by theorem (2.6) and proposition (3.3, (i)) it follows that $\prod_{i=1}^{n} T_i(V_i)$ is $\Lambda(\alpha)-$limited set in $\prod_{i=1}^{n} F_i$. Thus $\prod_{i=1}^{n} T_i$ is $\Lambda(\alpha)-$limited operator.

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