# Preopen and semi-open in $T_0$-Alexandroff space

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## الملخص

## "المجموعات تمهيد مفتوحة والمجموعات شبه مفتوحة في فضاء $T_0$- الكساندروف"

سابقا تمت دراسة خواص بعض المجموعات المفتوحة المعممة في فضاء $T_0$- الكساندروف والذي يحقق السلسلة الصاعدة(ACC) مثل المجموعة تمهيد مفتوحة و المجموعة شبه مفتوحة والمجموعة ألفا مفتوحة .....

في هذا البحث قمنا بتعميم النتائج السابقة علي أي فضاء $T_0$- الكساندروف سواء يحقق خاصية السلسلة الصاعدة (ACC) أو لا يحققها. وقمنا أيضا بتطبيق النتائج على مجموعات أخرى في الفضاء.

## ABSTRACT

Finite topological spaces became much more essential in topology،with the development of computer science and the need for digital topology. Finite spaces are a subclass of the class of $T_0$-Alexandroff spaces. The authors in [8] have studied properties of those finite spaces.

Elatrash and Mahdi in [9] have studied the properties of a more general class of spaces than finite spaces yet is a subclass of $T_0$-Alexandroff spaces whose corresponding poset satisfies the ascending chain condition (ACC). They have introduced a characterization of basic notions of open sets, such as preopen, semiopen, and alpha open set.

In the current paper we introduce notations, elementary facts and characterizations of notions of near openness along some of their properties in the class of $T_0$ Alexandroff spaces (Posets). We also apply the main results to some classes of spaces.

**Kay words:** $T_0$-Alexandroff space, preopen sets, semi-open sets, regular-open sets, $\alpha$-open sets.

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## INTRODUCTION:

An Alexandroff space *X* is a topological space in which arbitrary intersection of open sets is open or equivalently، arbitrary union of closed sets is closed. The class of Alexandroff spaces includes two important classes beside others namely; the class of locally finite spaces which in turn includes the class of all finite topological spaces.

Alexandroff spaces were first introduced in 1937 by Russian Mathematician P. S. Alexandrov under the name discrete spaces in [1] where he provided the characterizations in terms of sets and neighborhoods [17] .

The name discrete spaces later came to be used for topological spaces in which every subset is open and the original concept lay forgotten.

With the advancement of categorical topology in the 1980s، Alexandroff spaces were rediscovered when the concept of finite generation was applied to general topology and the name finitely generated spaces was adopted for them. Alexandroff spaces were also rediscovered around the same time in the context of topologies resulting from denotational semantics and domain theory in computer science.

In [14], M.C. McCord had observed that there was a duality between partially ordered sets and spaces which were precisely the $T_0$ versions of the spaces that Alexandroff had introduced. P Johnstone referred to such topologies as Alexandroff topologies in [11].

In [2], F. G. Arenas independently proposed this name for the general version of these topologies. It was also a well-known result in the field of modal logic that a duality exists between finite topological spaces and preorders on finite sets (the finite modal frames for the modal logic S4).

In [15], C. Naturman extended these results to a duality between Alexandroff spaces and preorders in general, providing the preorder characterizations as well as the interior and closure algebraic characterizations. A systematic investigation of these spaces from the point of view of general topology which had been neglected since the original paper by Alexandroff, was taken up by F.G. Arenas in [2].

Inspired by the use of Alexandroff topologies in computer science، applied mathematicians and physicists in the late 1990's began investigating the Alexandroff topology corresponding to causal sets which arise from a preorder defined on spacetime modeling causality.

The main example of an Alexandroff space is the poset ($P$, ≤) where $B = \{\uparrow x : x \in P\}$ is a basis for the topology. This topology - denoted by $\tau(\leq)$ – is a $T_0$ Alexandroff space. Conversely, if ($X$, τ) is an Alexandroff space, we may define a partial order (called the Alexandroff specialization order) $\leq_\tau$ on $X$ by setting $a \leq_\tau b$ if $a \in \overline{\{b\}}$.

The specialization order is reflexive and transitive. It turns out that it is antisymmetric- and hence partial order - if and only if $X$ is $T_0$. Moreover, if ($X$, ≤) is a poset and if $\tau(\leq)$ is its induced $T_0$- Alexandroff topology, then the specialization order of $\tau(\leq)$ is the order ≤ itself, i.e. $\leq_{\tau(\leq)} = \leq$. On the other hand, if ($X$, τ) is a $T_0$ Alexandroff space with specialization order $\leq_\tau$ then the induced topology by the specialization order is the original topology, i.e. $\tau(\leq_\tau) = \tau$ [2] . Therefore, $T_0$- Alexandroff spaces can be completely determined by their specialization orders.

We say that a poset ($X$, ≤) satisfies the ascending chain condition in short (*ACC*) if for every nondecreasing sequence $x_1 \leq x_2 \leq \ldots \leq x_i \leq \ldots$ in $X$ there exists $n$ such that $x_n = x_{n+1} = \ldots$ . A $T_0$- Alexandroff space whose corresponding poset satisfies the (*ACC*) is called *Artinian* $T_0$*- Alexandroff space*. It is known that the (*ACC*) is equivalent to *the maximal condition* (*MXC*); namely, every subset of a poset has a maximal element. In a similar way we define the *descending chain condition* (*DCC*) and also, it is known that (*DCC*) is equivalent to the *minimal condition* (*MNC*) namely, Every non-empty subset has a minimal element. The $T_0$- Alexandroff space in which the (*DCC*) holds is called *Noetherian* $T_0$ - Alexandroff space. If both (*ACC*) and (*DCC*) hold then the $T_0$- Alexandroff space will satisfy the finite chain condition denoted by (*FCC*). Finite topological spaces are locally finite spaces, and locally finite spaces are Artinian $T_0$- Alexandroff spaces.

In 1961, Levine in [12] introduced the notion of a semi-open set in any topological space. A subset $A$ in a topological space $X$ is called semi-open if and only if $A \subset \overline{int\, A}$. In 1982, Mashhour et. al. [13] defined the concept of a preopen set. A subset $A$ is preopen in $X$ if and only if $A \subset \mathrm{int}\,\overline{A}$. There are no implications between these two concepts. This means that semi-open sets need not be preopen and vice verca. In a finite topological space $X$, every preopen set in $X$ is semi-open. The converse may not be true. In fact, preopen sets in $X$ are semi-open sets in a larger class of $T_0$-Alexandroff spaces that satisfy the (*ACC*). However, in a general $T_0$ - Alexandroff spaces preopen sets may not be semi-open sets. The authors in [8] have studied properties of finite spaces.

Elatrash and Mahdi in [9] have studied the properties of a more general class of spaces than finite spaces yet is a subclass of $T_0$- Alexandroff spaces whose corresponding poset satisfies the ascending chain condition (*ACC*). They have introduced a characterization of basic notions of near open sets, such as preopen, semiopen, and alpha open sets.

Here, we characterize preopen, semiopen, $\alpha$-open sets in the general case of a $T_0$ - Alexandroff space, i.e., in any poset.

## Notation:

Let $A$ be subset of a $T_0$ - Alexandroff space $(X, \tau)$ and $x \in X$.

$A^c$ = complement of $A$.

$\uparrow x = \{y \in X : y \geq x\}$

$\uparrow A = \{y \in X : y \geq x, \forall x \in A\}$

$\downarrow x = \{y \in X : y \leq x\}$

$\downarrow A = \{y \in X : y \leq x, \forall x \in A\}$

If $(X, \tau)$ satisfies the (*ACC*) then:

$M$ = the set of maximal elements in $X$.

$M(A)$ = the set of maximal elements in $A$.

$\hat{x} = \uparrow x \cap M$.

$\hat{A} = \uparrow A \cap M$.

$\top$ = The maximum element of $X$ if any.

And if $(X, \tau)$ satisfies the (*DCC*) then:

$m$ = the set of minimal elements in $X$.

$m(A)$ = the set of minimal elements in $A$.

$\breve{x} = \downarrow x \cap m$.

$\breve{A} = \downarrow A \cap m$.

$\perp$ = The minimum element of $X$ if any.

## Definitions and elementary results:

**Definition 2.1** A subset $A$ of a space $(X, \tau)$ is called:

(1) a semi-open set if $A \subseteq \overline{A^\circ}$, and is called a semi-closed set if $A^c$ is semi-open. If $A$ is both semi-open and semi-closed, $A$ is called semi-regular.

(2) a preopen set if $A \subseteq \overline{A}^\circ$ and is called a preclosed set if $A^c$ is preopen,

(3) an α-open set if $A \subseteq \overline{A^\circ}^\circ$ and is called an α-closed set if $A^c$ is α-open.

(4) a regular open set if $A = \overline{A}^\circ$, and is called a regular closed set if $A^c$ is regular open.

(5) a semi- preopen set if $A \subseteq \overline{\overline{A}^\circ}$, and is called a semi- preclosed set if $A^c$ is semi- preopen.

It follows from the above definitions that:

- $A$ is a semi-closed if and only if $\overline{A}^\circ \subseteq A$,
- $A$ is a preclosed if and only if $\overline{A^\circ} \subseteq A$
- $A$ is an α-closed if and only if $\overline{\overline{A}^\circ} \subseteq A$
- $A$ is a semi- preclosed if and only if $\overline{A^\circ}^\circ \subseteq A$.

If $A$ is "semi-closed and semi-open" (resp. "preclosed and preopen", "α-closed and α-open") then $A$ is called semi-clopen (resp. preclopen, α-clopen)

In what follows, by X we always mean a topological space $(X, \tau)$. For each $A \subset X$, the closure (resp. interior, exterior, boundary) of $A$ will be denoted by $\overline{A}$ (resp. int $(A)$ or $A^\circ$, ext $(A)$, $\partial A$).

The set of all preopen sets of a space $(X, \tau)$ is denoted by PO$(X, \tau)$. The α-closure [ resp. semi-closure, preclosure, semi-preclosure] of $A \subseteq X$ is denoted by α- cl$(A)$ [ resp. scl$(A)$, pcl$(A)$, spcl$(A)$] is the smallest α- closed [resp. semi-closed, preclosed, semi-preclosed] set containing $A$. A point $x \in X$ is called a θ-closure point of $A$ if $A \cap$ cl$(V) \neq \phi$ for every open set $V$ containg $x$, the set of all θ - closure points of $A$ is called θ - closure of $A$ and is denoted by $cl_\theta(A)$.

**Definition 2.2.** A space $(X, \tau)$ is called Alexandroff space if the intersection of arbitrary family of open sets is open, or equivalently, the union of arbitrary family of closed sets is closed.

**Definition 2.3.** A space $(X, \tau)$ is called a locally finite space if every point $x \in X$ has a neighborhood that meets only finitely many open sets.

It follows that in an Alexandroff space $X$, for each point $x$, there is a smallest neighborhood which is contained in each other neighborhood of $x$. For each $x \in X$, let $U_x = \bigcap\{V : V \text{ is an open set containing } x\}$

Clearly $U_x$ is the smallest open set containing $x$ since $X$ is an Alexandroff space.

It is clear that all finite spaces are locally finite and all locally finite spaces are Alexandroff.

**Finite spaces $\Rightarrow$ Locally finite spaces $\Rightarrow$ Alexandroff spaces**

**Lemma 2.4.** The class $U = \{U_x : x \in X\}$ is a base for a finite space $(X, \tau)$. Each base for $\tau$ contains $U$.

Notice that if $X$ is an Alexandroff space, then $X$ is $T_1$ if and only if $U_x = \{x\}$. It follows that $X$ is discrete and so every point is an isolated point.

**Remark 2.5.** Observe that if $x$ and $y$ are two points in a space $X$, then $\boldsymbol{y} \in U_x$ if and only if $U_y \subseteq U_x$

**Definition 2.6.** [14]. In a $T_0$ - Alexandroff space, for two points $x, y \in X$, $x \leq y$ if $U_y \subseteq U_x$.

We will write $y \geq x$ to mean $x \leq y$ It follows from 2.5 that $U_x = \{y \in X: y \geq x\}$. Let us write $\uparrow x$ for $U_x$.

**Remark 2.7.** From Definition 2.6, the relation $\leq$ is reflexive and transitive since $\subseteq$ is so.

**Proposition 2.8.** In a space $X$, $x \leq y$ if and only if $x \in \overline{\{y\}}$.

**Proof.** Let $x \leq y$ and $x \neq y$. Then $y \in U_x$ which is the smallest open set containing $x$. Then for any open set $G$ containing $x$, we have $(G \setminus \{x\}) \cap \{y\} \neq \phi$. This means $x$ is an accumulation point of $y$. Therefore $x \in \overline{\{y\}}$. Conversely, let $x \in \overline{\{y\}}$ then $G \cap \{y\} \neq \phi$ for every open sets $G$ containing $x$. So $y \in G$ for every open set $G$. Take $G = U_x$. By Remark 2.5, we get $U_y \subseteq U_x$. This shows that $x \leq y$. □

**Proposition 2.9.** Let $(X, \tau)$ be a $T_0$- Alexandroff space, and let $A \subseteq X$. Then the following hold:

(1) For $x \in X$, $\overline{\{x\}} = \downarrow x$.

(2) $A^{\circ} = \{x \in A: \uparrow x \subseteq A\} = \cup\{\uparrow x : \uparrow x \subseteq A\}$

(3) $\overline{A} = \cup\{\downarrow x : x \in A\}$.

(4) $A' = \{x \in \overline{A} : \exists y \in A \ with \ x < y\}$

**Proof.** (1) It is straight forward By Proposition 2.8.

(2) By definition $x \in A^\circ$ iff there exists an open set $U$ such that $x \in U \subseteq A$ iff

$x \in U_x \subseteq U \subseteq A$ iff $x \in U_x \subseteq A$ iff $x \in \uparrow x \subseteq A$.

(3) $\overline{A} = \overline{\cup\{x: x \in A\}} = \cup\{\overline{\{x\}} : x \in A\} = \cup\{\downarrow x : x \in A\}$. The second equality holds since the space is Alexandroff space.

(4) $x \in A'$ iff for every open set $U$, $x \in U$ we have $U \cap A\backslash\{x\} \neq \phi$ iff $U_x \cap A\backslash\{x\} \neq \phi$ iff $\uparrow x \cap A \backslash \{x\} \neq \phi$ iff $\exists y \in A$, with $x < y$ iff $x \in \overline{A}$ and $\exists y \in A$ with $x < y$. □

In the following theorem we summarize some of the properties of preopen sets.

**Theorem 2.10.** [6] For a subset $A$ of a topological space $(X, \tau)$ the following
conditions are equivalent:

(1) $A$ is preopen.

(2) The semi-closure of $A$ is a regular open set.

(3) $A$ is the intersection of an open set and a dense set.

(4) $A$ is the intersection of a regular open set and a dense set.

(5) $A$ is a dense subset of some regular open subspace.

(6) $A$ is a dense subset of some open subspace.

(7) $A$ is a dense subset of some preopen subspace.

(8) $A$ is a preneighborhood of each one of its points.

(9) $scl(A) = Int(cl(A))$

(10) There exists a regular open set $R$ containing $A$ such that $cl(A) = cl(R)$.

## Here are the most fundamental properties of preopen sets:

•Noiri's Lemma [17]: If $A$ is semi-open and $B$ is preopen, then $A \cap B$ is semi-open in $B$ and preopen in $A$.

• Jankovi'c and Reilly's Lemma [5]: Every singleton is either preopen or nowhere dense.

•Arbitrary union of preopen sets is preopen.

•Finite intersection of preopen sets need not be preopen.

•The intersection of a preopen set and an $\alpha$- open set is a preopen set.

•The intersection $P \cap R$ of a preopen set $P$ and a regular closed (resp. regular open) set $R$ is regular closed (resp. regular open) in the preopen subspace $P$.

•A set is $\alpha$-open if and only if it is semi-open and preopen.

•A set is clopen if and only if it is closed and preopen.

•A set is open if and only if it is locally closed and preopen if and only if it is $A$- set and preopen if and only if it is a $B$-set and preopen.

•A set is regular open if and only if it is semi-closed and preopen.

•If $U$ is a preopen subspace of a space $(X, \tau)$ and $V$ a preopen subset of $(U, \tau|U)$, then $V$ is preopen in $(X, \tau)$.

•If $V$ is preopen such that $U \subseteq V \subseteq Cl(U)$, then $U$ is also preopen.

•If $V$ is preopen such that $V \subseteq U \subseteq X$, then $V$ is also preopen in $(U, \tau|U)$

•If A is an $\alpha$-open subset of a space $(X, \tau)$ then a subset $U$ of $A$ is preopen in $(A,\tau|A)$ if and only if $U$ is preopen in $(X, \tau)$.

•If $A$ is a preopen subset of a space $(X, \tau)$, then for every subset $U$ of $A$ we have $A \cap scl(U) = scl_A(U)$.

•If $A$ is a preopen subset of a space $(X, \tau)$, then for every subset $U$ of $A$ we have $Int_A(cl_A(U)) = A \cap Int(cl(U)$.

•If $P$ is preopen and $S$ is semi-open, then $P \cap cl(S) = cl( P\cap Int(S)) = cl(P \cap S) = cl(P \cap cl(S)) = cl(Int(cl(P) \cap S))$.

•If $A$ is preopen, then $Cl(A) = Cl_\theta(A)$.

•$PO(X, \tau) = PO(X, \tau^\alpha)$.

•Let $(X_i)_{i \in I}$ be a family of spaces and $\phi \neq A_i \subseteq X_i$ for each $i \in I$. Then, $\prod_{i \in I} A_i$ is preopen in $\prod_{i \in I} X_i$ if and only if $A_i$ is preopen in $X_i$ for each $i \in I$ and $A_i$ is non-dense for only finitely many $i \in I$.

## Characterization of Preopen sets:

In [9] the authors characterized the class of preclosed sets in the class of Artinian $T_0$-Alexandroff spaces as the subsets that contain $\downarrow x$ for all points $x \in A \cap M$.

In the next theorem we extend the characterization to the case of $T_0$-Alexandroff spaces rather than the class of Artinian $T_0$-Alexandroff spaces. Therefore, this characterization holds in every poset. In fact it states that a set $A$ is preclosed in any poset if it contains $\downarrow x$ whenever $\uparrow x \subseteq A$. From which it follows that the subsets that have no interior are always preclosed.

**Theorem 3.1.** A subset $A$ is preopen of a $T_0$-Alexandroff space $(X, \tau)$ if and only if $\forall x \in A^c$, if $\downarrow x \cap A \neq \phi$ then $\uparrow x \cap A \neq \phi$.

**Proof.** $(\Rightarrow)$ Suppose that $A$ is a preopen subset of a space $(X, \tau)$. i.e., $A \subseteq \overline{A}^{\circ}$. Assume that $\downarrow x \cap A \neq \phi, \forall x \in A^c$. Then for every $x \in A^c, \exists y_x \in \downarrow x \cap A$. i.e., $y_x \in A$ and $y_x \in \downarrow x$. Since $y_x \in A$ then by hypothesis $y \in \overline{A}^{\circ}$ then $\uparrow y_x \subseteq \overline{A}$ but $x \in \uparrow y_x$. It follows that $x \in \overline{A}$ then $\exists x_\circ \in A$ and $x \in \downarrow x_\circ$ this implies $x_\circ \in \uparrow x$ then $x_\circ \in \uparrow x \cap A$.

$(\Leftarrow)$ Suppose that $\forall x \in A^c, \downarrow x \cap A \neq \phi \Rightarrow \uparrow x \cap A \neq \phi$. Assume that $A$ is not preopen. Then $\exists y \in A, y \notin \overline{A}^{\circ}$ i.e. $y \in A$ and $\uparrow y$ is not a subset of $\overline{A}$. It follows that $\exists y_\circ \in \uparrow y, y_\circ \notin \overline{A}$. Consequently, $\forall z \geq y_\circ, z \notin A$. Then $\uparrow y_\circ \cap A = \phi$, while $\downarrow y_\circ \cap A \supseteq \{y\} \neq \phi$. This contradicts the hypothesis that if $\downarrow y_\circ \cap A \neq \phi$ then $\uparrow y_\circ \cap A \neq \phi$. □

**Corollary 3.2.** If $\forall x \in A^c, \downarrow x \cap A = \phi$ then $A$ is preopen.

**Proof.** Clearly, if $\forall x \in A^c, \downarrow x \cap A = \phi$ holds then $\forall x \in A^c, \downarrow x \cap A = \phi \Rightarrow \uparrow x \cap A \neq \phi$ holds. □

**Definition 3.3**. A subspace $A$ of a topological space $X$ is called *disconnected* iff there are disjoint nonempty open sets $U_1$ and $U_2$ such that $A = U_1 \cup U_2$, $A$ is *connected* if $A$ is not disconnected. If $A$ is the maximal connected subspace then $A$ is called *connected component*.

Corollary 3.2 proves that every connected component $A$ of any $T_0$-Alexandroff space is preopen, since for any connected component $A$ the condition $\forall x \in A^c, \downarrow x \cap A = \phi$ holds [9].

The implication statement in Theorem 3.1 can be restated as follows:

A subset $A$ of a $T_o$- Alexandroff space ($X$, $\tau$) is preopen if and only if $\forall x \in A^c$ if $\uparrow x \cap A = \phi$ then $\downarrow x \cap A = \phi$.

**Corollary 3.4.** A subset $A$ is preclosed of a $T_0$-Alexandroff space ($X$, $\tau$) if and only if $\forall x \in A$, if $\downarrow x \cap A^c \neq \phi$ then $\uparrow x \cap A^c \neq \phi$ . Equivalently, A is preclosed if and only if $\forall x \in A$, if $\uparrow x \subseteq A$ then $\downarrow x \subseteq A$.

**Proof**. Apply Theorem 3.1 for $A^c$.

**Example 3.5.**

Let $X = Z$ "the set of integers" with the natural partial order. Let $A$ be the set of all even numbers (see Figure 1). Then by Theorem 3.1 $A$ is preopen, since the condition $\forall x \in A^c$, if $\downarrow x \cap A \neq \phi$ then $\uparrow x \cap A \neq \phi$ holds. We know that it is preopen since $\overline{A} = Z$ from which it follows that $\overline{A}^{\circ} = Z$, which implies that $A \subseteq \overline{A}^{\circ}$ , i.e., preopen. In fact the condition holds for the complement; the odd numbers, therefore, $A$ is preclopen.

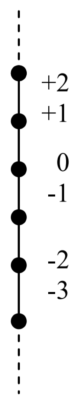


**Figure 1**

**Theorem 3.6**. Let ($X$, $\tau$) be a $T_0$-Alexandroff space. Let $D$ be a subset of $X$. Then $D$ is dense iff for any $x \in X$, there exists $d \in D$ such that $x \leq d$.

**Proof.** Let $D$ be a dense subset of $X$. Let $x \in X$ be arbitrary point. It follows that $x \in \overline{D}$ It follows that there exists $d \in D$ such that $x \leq d$. Conversely, for any $x \in X$ we have $x \leq d$ for some $d \in D$. It follows that $x \in \overline{\{d\}} \subseteq \overline{D}$. Therefore, $X \subseteq \overline{D}$. □

Theorem 3.5 states that dense subsets must contain upper bounds of *X*. It follows that in any $T_0$-Alexandroff space if a set of maximals *M* exists, then *M* and all supersets of *M* are dense.

**Example 3.7.** Let $X = \{a, b, c, d, e, f\}$ and let $A = \{a, b, c, f\}$. Then A is not preopen, since $e \in A^c$ with $\downarrow e \cap A \neq \phi$ while $\uparrow e \cap A = \phi$. (See figure2)

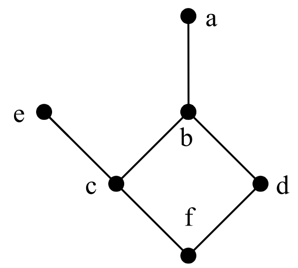


**Figure 2**

Here, we show that the characterization in case of Artinian $T_0$-space follows from Theorem 3.1. These results are in [9].

**Corollary 3.8.** Let $(X, \tau)$ be an Artinian $T_o$- Alexandroff space. Then the set *A* is preclosed if and only if $\downarrow x \subseteq A$ for all $x \in A \cap M$.

**Proof**. Suppose that *A* is preclosed, It follows by Theorem 3.1 that $\forall\, x \in A$, if $\uparrow x \subseteq A$ then $\downarrow x \subseteq A$. Since $\{x\} = \uparrow x$ for all $x \in M$, then $\uparrow x \subseteq A$ holds for those $x \in A \cap M$. It follows that, $\downarrow x \subseteq A$ for all $x \in A \cap M$ holds.
Conversely, suppose that $\downarrow x \subseteq A$ for all $x \in A \cap M$ holds. Let x be any element of *A* with $\uparrow x \subseteq A$. Then $\uparrow x \cap M \subseteq A \cap M$. Let $a \in \uparrow x \cap M$. By hypothesis $\downarrow a \subseteq A$.
Since $x \leq a$ then $\downarrow x \subseteq \downarrow a$. It follows that $\downarrow x \subseteq A$. Therefore, by Theorem 3.1 *A* is preclosed. □

**Corollary 3.9.** Let $(X, \tau)$ be an Artinian $T_o$- Alexandroff space.Then:

a) the set *A* is preopen if and only if $\downarrow x \cap A = \phi$ for all $x \in A^c \cap M$. Equivalently, *A* is preopen if and only if $\hat{x} \subseteq A$ for all $x \in A$.

b) If *X* contains a top element $\top$, then a nonempty subset *A* is preopen if and only if $\top \in A$ if and only if *A* is dense.

**Proof.** Both (a), (b) are consequences of Corollary 3.8.
It follows that we can determine all preclosed subsets in case the poset has a maximal element $\top$; if $\top \in A$ then *A* is preopen iff *A* is the whole poset, and

if $\notin A$ then $A$ is always preclosed. It follows that, in this case, preopen sets are the empty set and all the ones containing , in fact in this case a set $A$ is preopen iff $A$ is dense. And clearly, in this case the class of preopen sets form a topology. In fact this topology is the topology of the class of $\alpha$-open sets $\tau_\alpha$.

**Theorem 3.10.** Let $(X,\tau)$ be a $T_0$-Alexandroff space and let $S$ be a subset of $X$. Then:

(1) $pint(S) = S \cap \mathrm{int}(cl\ S)$

(2) $pcl(S) = S \cup cl(int\ (S))$

**Proof.** For the proof see [6]. □

**Theorem 3.11.** Let $(X, \tau)$ be a $T_0$-Alexandroff space and let $S$ be a subset of $X$. Then

1) $pint(S) = \{x : x \in S$ and $\uparrow y \cap S \neq \phi, \forall y \geq x\}$

2) $pcl(S) = \{x : x \in S$ or $\uparrow y \subseteq S$ for some $y \geq x\}$

**Proof.**

1) $x \in pint(S)$ iff $x \in S$ and $x \in int(cl(S))$ iff $x \in S$ and $\uparrow x \subseteq cl(S)$ iff $x \in S$ and $\forall y \in \uparrow x, y \in cl(S)$ iff $x \in S$ and $\forall y \in \uparrow x$ , $y \in \downarrow z$ for some $z \in S$ iff $x \in S$ and $\forall y \geq x$, $\exists\ z \in S$ and $z \geq y$ iff $x \in S$ and $\forall y \geq x$ , $z \in \uparrow y \cap S$ i.e., $\uparrow y \cap S \neq \phi$.

2) $x \in pcl(S)$ iff $x \in S$ or $x \in cl(int\ S)$ iff $x \in S$ or $x \in \downarrow y$ for some $y \in int(S)$ iff $x \in S$ or $x \in \downarrow y$ , $\uparrow y \subseteq S$ for some $y \in S$ iff $x \in S$ or $\uparrow y \subseteq S$ for some $y \geq x$. □

In order to find $pcl(S)$ we look for all those $y$ in $S$ such that $\uparrow y \subseteq S$ and then we take $\downarrow$y for $pcl(S)$. Clearly, this shows that $pcl(S) \subseteq cl(S)$, since $cl(S) = \downarrow S$.

**Example 3.12.** $X = \{ c, d, e, f, g , a_0, a_1, a_2, \ldots \}$ and let $A = \{e, g, a_0, a_1, a_2, \ldots\}$. Then $pint(A) = \{a_0, a_1, a_2, \ldots\}$ and $pcl(A) = \{ c, d, e, g, a_0, a_1, a_2, \ldots\}$. (See figure 3)

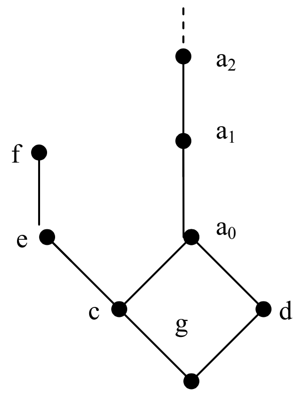

**Figure 3**

**Corollary 3.13.** Let $(X, \tau)$ be an Artinian $T_o$- Alexandroff space and $A$ is a subset of $X$. Then**:**

a) $pint(S) = \{x \in S: \hat{x} \subseteq S\}$

b) $pcl(S) = S \cup \{\downarrow z: z \in S \cap M\}$

**Proof.**

a) By 3.11 (1) $pint(S) = \{ x : x \in S$ and $\uparrow y \cap S \neq \phi, \forall y \geq x\}$. Note that $\hat{x} \geq x$, $\uparrow \hat{x} = \hat{x}$. It follows that if $x \in pint(S)$ then $x \in S$ and $\hat{x} \subseteq S$. Conversely, if $x \in S$ and $\hat{x} \subseteq S$, then $\hat{x} \in \uparrow y \cap S$. Therefore, $\uparrow y \cap S \neq \phi$, $\forall y \geq x$, then $x \in pint(S)$.

b) By 3.11 (2) $pcl(S) = \{ x : x \in S$ or $\uparrow y \subseteq S$ for some $y \geq x\}$. For any $x \in \downarrow z$ with $z \in S \cap M$, let $y = z$ then $y \geq x$, $\uparrow y \subseteq S$. It follows that $x \in pcl(S)$. Conversely, if $\uparrow y \subseteq S$ for some $y \geq x$, then let z be any point in $\hat{y} = \uparrow y \cap M$. We know that $\uparrow y \cap M \subseteq S \cap M$. It follows that $x \in \downarrow z$, $z \in S \cap M$. □

**Example 3.14.** Let $X = \{a, b, c, d, e, f, g\}$ and let $A = \{a, e, c, g\}$. Then $pint(A) = \{a\}$ and $pcl(A) = \{a, b, c, d, e, g\}$. (See figure 4)

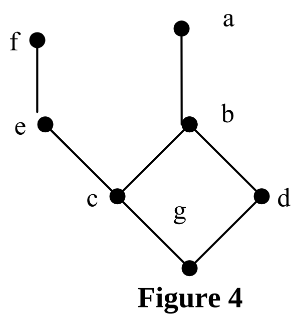


**Figure 4**

## Characterization of semi-open sets:

In the next Theorem 4.1 we characterize the class of semiopen subsets in an arbitrary poset. It states that a subset $A$ of a poset $X$ is semiopen iff for every $x \in A$ there is $y \in A$ such that $\uparrow y \subseteq A$.

**Theorem 4.1.** Let $(X, \tau(\leq))$ be a $T_o$- Alexandroff space. A subset $A$ is semi-open if and only if $\forall x \in A, \exists\, y \in A, x \leq y, \uparrow y \subseteq A$.

**Proof**. Assume $A \subseteq \overline{A^\circ}$ and let $x \in A$, then by hypothesis $x \in \overline{A^\circ}$ then $\exists\, y \in A^\circ \subseteq A$ such that $x \in \downarrow y$, and $\uparrow y \subseteq A$. Therefore, $\forall x \in A, \exists y \in A, x \leq y, \uparrow y \subseteq A$, Conversely, let $x \in A$, then by hypothesis $\exists y \in A, x \leq y, \uparrow y \subseteq A$. It follows that $y \in A^\circ$. Since $x \in \downarrow y$, $y \in A^\circ$, then $x \in \overline{A^\circ}$. □

Theorem 4.1 says that a set $A$ is semi-open iff $\uparrow x$ is "finally" in $A$ for every $x \in A$.

**Example 4.2.** Let $X = \{c, d, e, f, g, a_0, a_1, a_2, \ldots\}$ and let $A = \{c, d, g, a_0, a_1, a_2, \ldots\}$. Then $A$ is semi-open. However, the set $B = \{c, d, e, g, a_0, a_1, a_2, \ldots\}$ is not semi-open. Let $E = \{a_i : i \text{ is even}\}$ $E$ is pre-open but not semi-open. (see figure 5)

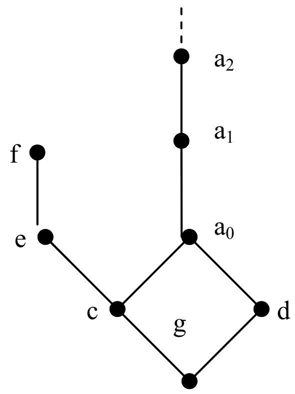


**Figure 5**

Theorem 4.1 generalizes the characterization of semi-open sets in the Artinian $T_0$- Alexandroff spaces to all $T_0$-Alexandroff spaces.

**Corollary 4.3.** Let $(X, \tau)$ be an Artinian $T_o$-Alexandroff space, then a set $A$ is semi-open if and only if $M(A) \subseteq M$.

**Proof**. Suppose that $A$ is semi-open. Then by Theorem 4.1, we have $\forall x \in A, \exists\, y \in A, \ x \leq y, \uparrow y \subseteq A$. Let $x \in M(A)$, then there exists $y \in A$ such that
$\uparrow y \subseteq A$. Since $X$ is maximal in $A$ then y = x. It follows that $\uparrow x \subseteq A$. Then

$\uparrow x \cap M \subseteq A \cap M$ , i.e., $x = \hat{x} \subseteq A \cap M$ . Therefore, $x \in M$ . It follows that $M(A) \subseteq M$.

Conversely, Suppose that $M(A) \subseteq M$. Clearly, for any $x \in A$ there is $y \in M(A)$ such that $x \leq y$. Since $M(A) \subseteq M$, then $\uparrow y = y \in A$, it follows that $\forall x \in A, \exists y \in A, x \leq y, \uparrow y \subseteq A$. □

**Example 4.4.** Let $X = \{a, b, c, d, e, f\}$, $A = \{a, b\}$, then $A$ is not semi-open, while $B = \{e\}$, $C = \{d, e\}$ are semi-open. (See Figure 6 )

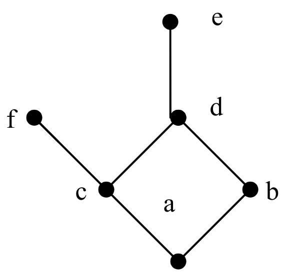


**Figure 6**

**Corollary 4.5.** A subset $A$ is semi-closed of a $T_0$-Alexandroff space $(X, \tau)$ if and only if $\forall x \in A^c, \exists y \in A^c, x \leq y, \uparrow y \subseteq A^c$.

**Theorem 4.6.** A subset $A$ is regular open of a $T_0$-Alexandroff space $(X, \tau)$ if and only if $\forall x \in A^c, \exists y \in A^c , y \geq x$ such that $\uparrow y \cap A = \phi$ and $\downarrow y \cap A = \phi$ .

**Proof**. ($\Rightarrow$) Let $A$ be a regular open subset of $X$ then $A \subseteq \overline{A}^{\circ}$ and $\overline{A}^{\circ} \subseteq A$ i.e., $A$ is both preopen and semi-closed. Let $x \in A^c$, since $A$ is semi-closed then $\exists\, z \in A^c$ , $z \geq x$ such that $\uparrow z \cap A = \phi$ and since $A$ is preopen then $\downarrow z \cap A = \phi$.

($\Leftarrow$) Assume $A \neq \overline{A}^{\circ}$ ,then one of two cases may happen either $\exists a \in A \setminus \overline{A}^{\circ}$ or $\exists\, b \in \overline{A}^{\circ} \setminus A$.

<u>**Case 1:**</u> If $\exists a \in A$, $a \notin \overline{A}^{\circ}$ then $\uparrow$a is not a subset of $\overline{A}$. It follows that $\exists\, c \in \uparrow a$ with $c \notin \overline{A}$.Consequently, $c \notin A$. I.e., $c \in A^c$. It follows, by hypothesis, that $\exists\, y \in A^c$ and $y \geq c$ with $\uparrow y \cap A = \phi$ and $\downarrow y \cap A = \phi$. However , $a \in \downarrow y$ (since $y \geq c \geq a$ ) then $a \notin A$. This is a contradiction to the fact that $a \in A$.

<u>**Case 2**</u>**:** If $\exists b \in \overline{A}^{\circ}$ , $b \notin A$, then $b \in A^c$ . By hypothesis $\exists y \in A^c$ , $y \geq b$ such that

$\uparrow y \cap A = \phi$ and $\downarrow y \cap A = \phi$. Since $b \in \overline{A}^{\circ}$ then $\uparrow b \subseteq \overline{A}$. Since $y \in \uparrow b$ then $y \in \overline{A}$. It follows that $\exists c \in A$ such that $y \in \downarrow c$ . So $\uparrow y \cap A \neq \phi$. This is a contradiction to the fact that $\uparrow y \cap A = \phi$. □

**Example 4.7.** Let $X = \{a, b, c, d, e, f, g\}$ , $A = \{a, b, c\}$ then $A$ is not regular. (See figure 7).

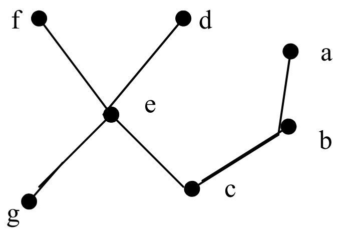


**Figure 7**

**Example 4.8.** Let $X = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c\}$ then $A$ is regular. (See figure 8)

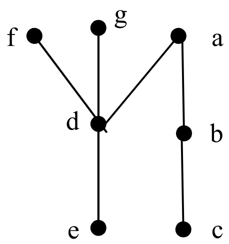


**Figure 8**

**Corollary 4.9.** A subset $A$ is regular closed of a $T_o$-Alexandroff space $(X, \tau)$ if and only if $\forall x \in A, \exists\, y \in A, y \geq x$ such that $\uparrow y \subseteq A$ and $\downarrow y \subseteq A$.

The family of all semi-open (resp. preopen, $\alpha$- open) is denoted by $SO(X)$ (resp. $PO(X)$, $\tau_\alpha$). Njåstad [16] proved that $\tau_\alpha$ is a topology on $X$. In general, $SO(X)$ and $PO(X)$ need not be topologies on $X$. A set $A$ is preopen [10] if and only if $A = U \cap D$ where $U$ is an open set and $D$ is a dense set. In [6], it has been shown that a set is $\alpha$-open if and only if it is semi-open and preopen.

**Theorem 4.10.** [4] Let $(X, \tau)$ be any topological space and let $S$ be a subset of $X$. Then:

1) $sint(S) = S \cap cl(int\, S)$

2) $scl(S) = S \cup int(cl\,(S))$

3) $int_{\alpha}(S) = S \cap int(cl(intS))$

**Theorem 4.11.** Let $(X, \tau)$ be a $T_0$-Alexandroff space and let $S$ be a subset of $X$. Then

1) $sint\,(S) = \{x: x \in S$ and $\uparrow y \subseteq S$ for some $y \geq x\}$

2) $scl\,(S) = \{x: x \in S$ or $\uparrow y \cap S \neq \phi, \forall\, y \geq x\}$

3) $int_{\alpha}(S) = \{x: x \in S$ and $\forall y \geq x, \exists\, z \in S, y \in \downarrow z, \uparrow z \subseteq S\}$

**Proof**.

(1) $x \in sint(S)$ iff $x \in S$ and $x \in cl(int\, S)$ iff $x \in \downarrow y$ for some $y \in int\,(S) \subseteq S$, iff $x \in S$ and $x \in \downarrow y$, $\uparrow y \subseteq S$ for some $y \in S$.

(2) $x \in scl(S)$, iff $x \in S$ or $x \in int(cl(S))$, iff $\uparrow x \subseteq cl(S)$ iff $\forall y \in \uparrow x$, $y \in cl(S)$ iff $y \in \uparrow x$, $y \in \downarrow z$, for some $z \in S$, iff $\uparrow y \cap S \neq \phi$, $\forall\, y \geq x$.

(3) $x \in int_{\alpha}(S)$ iff $x \in S$ and $x \in int(cl(int\, S))$ iff $\uparrow x \subseteq cl(int\, S)$ iff $\forall\, y \in \uparrow x$, $y \in cl(\text{int}\, S)$ iff $y \in \downarrow z$ for some $z \in int(S)$ iff $\uparrow z \subseteq S$.

Although the characterization for $pint(A)$ involves points $y \in X$, but we do not have to check it for all $y \in X$, in fact we need to check the condition only for those $y$ above $x$ only.

Theorem 3.11 is the natural generalization for the characterization Theorem 4.13, in [9]

**Equivalently, we can restate 3.11 [1] as follows:**

Let $(X, \tau)$ be a $T_0$-Alexandroff space and let $S$ be a subset of $X$, then $x \in pint(S)$ iff $\uparrow y \cap S \neq \phi$, $\forall\, y \geq x$.

**Corollary 4.12.** Let $(X, \tau)$ be an Artinian $T_0$ - Alexandroff space and $A$ is a subset of $X$. Then

(a) $pint(A) = \{x \in A : \hat{x} \subseteq A\}$

(b) $sint(A) = \{x \in A : \hat{x} \cap A \neq \phi\}$

(c) $pcl(A) = A \cup \{\downarrow x : x \in A \cap M\}$

(d) $scl(A) = A \cup \{x : \hat{x} \subseteq A\}$.

**Corollary 4.13.** Let $(X, \tau)$ be an Artinian $T_o$- Alexandroff space, and let $A$ be a subset of $X$, then

(i) $scl(A) \subseteq pcl(A)$.

(ii) $pint(A) \subseteq sint(A)$

## REFERENCE:

1. Alexandroff, P.(1937): Diskrete Räume, Mat. Sb.(N.S.) 2, 501-518.
2. Arenas, F.G.(1997): Some results on Alexandroff space, Preprint, Topology Atlas.
3. Arenas, F.G.(1999): Alexandroff spaces, Acta Math. Univ. Comenianae, 68, 17-25.
4. Andrijevic. D. (1984) : Some properties of the Topology of $\alpha$-sets, Mat. Vesnik 36, no. 1, 1-10.
5. janković D. and I. L. Reilly.(1985) : on semi-separation properties, Indian *J. Pure Appl. Math*., 16, 957-964.
6. Dontchev, J. Survey on preopen sets, The Proceedings of the (1998): Yatsushiro Topological conference, 1-18.
7. Dontchev, J. Topological properties defined in terms of generalized open sets, Presented at the (1997):Yatsushiro Topological Conference,Japan,1-14.
8. El-Atik,A.E., Abd El-Monsef,M.E and Lashin, E.I.(2001): On finite $T_o$ topological spaces, Proceedings of the ninth Prague Topological Symposium, (Prague), 75-90, Topological Atlas, Toronto, 2002.
9. Elatrash, M. and Mahdi, H.(2005): On $T_0$-Alexandroff spaces Journal of The Islamic University of Gaza, Series of Natural Studies \& Engineering, Vol. 13, No. 2, pp. 19-46.
10. Ganster, M.(1987) : Preopen sets and resolvable spaces, Kyungpook Math. J. 27(2), 135-143.
11. Johnstone, P.T.(1982,1986): Stone spaces, Cambridge University Press edition.
12. Levine, N.(1961): A decomposition of continuity in topological spaces,Amer. Math. Monthly 68, 44-46. MR 23 #A3548
13. Mashhour, A.S. , Abd El-Monsef, M.E. and El-Deep، S.N.(1982, 1983) : On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, no.53, 47-53. MR 87c:54002
14. McCord, M. C.(1966): Singular homology and homotopy groups of finite topological spaces, Duke Math. Jour، 33, 465-474.
15. Naturman, C.A.(1991): Interior Algebras and Topology, Ph.D. thesis, University of Cape Town Department of Mathematics.
16. Njåstad, O.(1965) : On some classes of nearly open sets, Paci c J. Math. 15, 961-970.

17. T. Noiri.(1984) :, Hyperconnectedness and preopen sets Rev.Roumaine Math. Pure Appl. 29 (4), 329-334.