Al-Aqsa University Journal (Natural Sciences Series),Vol.22, Pages 36 -62, Jan.2020
ISSN 2070-3155

SOME PROPERTIES RELATING TO GEGENBAUER MATRIX POLYNOMIALS IN TWO VARIABLES

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ملاحظة على مصفوفة متعددة الحدود جيجينباور في متغيرين

الملخص

يتناول هذا البحث دراسة نوع جديد من مصفوفة جيجينباور متعددة الحدود لمتغيرين. تم إيجاد كلا من المصفوفة التفاضلية للعلاقات التكرارية، مصفوفة المعادلات التفاضلية الجزئية لمتغيرين لمصفوفة جيجينباور متعددة الحدود والدالة المولدة المزدوجة لتتابع الدوال. كذلك بالإضافة الى ثنائي الخط للدالة المولدة المزدوجة لتتابع الدالة، خاصية خاصة لمصفوفة جيجينباور واستعراض مصفوفة جيجينباور متعددة الحدود لمتغيرين كمتسلسلة في مصفوفة هيرمايت متعددة الحدود.

Abstract

This paper deals with the study of some properties of the new kind of Gegenbauer matrix polynomials of two variables. The matrix differential recurrence relations, partial matrix differential equation of two variable of Gegenbauer matrix polynomials and double generating matrix functions are established. Furthermore, a bilinear double generating matrix function, a special property and expansion of two variables of Gegenbauer matrix polynomials as series of Hermite matrix polynomials has been presented.

Keywords: hypergeometric matrix function, Gegenbauer matrix partial differential equation, differential matrix recurrence relation, Hermite matrix polynomials of two variables.

Mathematics Subject Classifications (2016): Primary $33C45, 33C50$, $33A22$, $33E20$.

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1. Introduction: The study of special matrix functions is an independent field. Some results in the theory of classical orthogonal polynomials have been studied and extended to orthogonal matrix polynomials see for instance $[10,13,14,15,21,11]$. Laguerre, Legendre, Hermite, Gegenbauer and Jacobi matrix polynomials, have been introduced and studied in $[2,3,5,8,9]$ for matrix in $C^{N\times N}$ · The hypergeometric matrix functions with matrix differential equation and the explicit closed form general solution of it has been given and studied in $[4,6,7]$. Throughout this paper, if $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z . which are defined in an open set Ω of the complex plane and if A is a matrix in $C^{N\times N}$ such that $\sigma(A)\subset\Omega,$ then from the properties of the matrix functional calculus $[9,16]$, it follows that $f(\mathrm{A})\,\mathrm{g}(\mathrm{A}) = g(\mathrm{A})\,f(\mathrm{A})$. If A is a matrix with $\sigma(\mathrm{A})\subset \mathrm{D}_0$, where D_0 is the complex plane cut along the negative real axis and $\log(\mathrm{z})$ denotes the principal logarithm of z, then $z^{1/2}$ represent $\exp(\frac{1}{2}\log(\mathrm{z}))$, then $A^{1/2}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A $[2,3]$. For $\|A\|$ denotes any matrix norm for which $\|I\|=1$. If $\|A\|<1$ for a matrix A in $C^{N\times N}$, then $(I-A)^{-a}$, where a is a positive integer, exists and given by $[17]$

$$(I-A)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n A^n}{n!} \tag{1}$$

Also, consider the complex space $C^{N\times N}$ of all square complex matrices of common order N .
For a positive stable matrix A in $C^{N\times N}$, if $\mathrm{Re}(\lambda)>0$ for all $\lambda\in\sigma(A)$,where $\sigma(A)$ denotes the set of all eigenvalues of A. If A_0, A_1, A_2 , ... , A_n are elements of $C^{N\times N}$ and $A_n\neq 0$, then we call $P_n(x)=A_n x^n + A_{n-1}x^{n-1}+\ ...\ +A_1 x\ +A_0$ a matrix polynomials of degree n in x. If $A+nI$ is invertible for every integer $n\geq 0$, then from $[6]$ the matrix version of the pochhammer symbol is

$$(A)_n = \frac{\Gamma(A+nI)}{\Gamma(A)}. \tag{2}$$

$$(A)_n = \begin{cases} I & if\ n=0 \\ A(A+I)(A+2I)\ldots(A+(n-1)I) & if\ n=1,2,\ldots \end{cases}, \tag{3}$$

from (3), it is easy to find that

$$(A)_{n-r} = \frac{(-1)^r (A)_n}{(I - A - nI)_r}; \qquad 0 \le r \le n. \tag{4}$$

From $[22, pp.58]$, one obtains in $[7]$

$$\frac{(-1)^r}{(n-r)!} I = \frac{(-n)_r}{n!} I = \frac{(-nI)_r}{n!}, \quad 0 \le r \le n. \tag{5}$$

If A and B are members of $C^{N\times N}$ for which $AB = BA$ and if, for all integer $n \ge 0$ $A + nI$, $B + nI$, and $(A + B + nI)$ are all invertible, where I is identity matrix in $C^{N\times N}$, then the beta matrix function $\beta(A, B)$ is defined by $[6]$

$$\beta(A, B) = \Gamma(A)\Gamma(B)\Gamma^{-1}(A + B). \tag{6}$$

Also if the matrices A, B and C in $C^{N\times N}$ for which $C + nI$ is invertible for all integer $n \ge 0$, the hypergeometric matrix function ${}_2F_1(A, B; C; x)$ has been given in the form $[6]$

$${}_2F_1(A, B; C; x) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n \ x^n}{(C)_n \ n!}, \tag{7}$$

it convergence for $|x| < 1$.

For any matrix A in $C^{N\times N}$ the following relation will exploit due to $[6]$

$$(1 - x)^{-A} = {}_1F_0(\mathrm{A}; -; x) = \sum_{\mathrm{n}=0}^{\infty} \frac{(A)_n \ \mathrm{x}^{\mathrm{n}}}{n\ !}; \qquad |x| < 1, \tag{8}$$

where $(A)_n$ denotes the pochhammer symbol given by (3). Also, we have the expansion of multinomial formula

$$(1 - u - v)^{-A} = \sum_{\mathrm{n}=0}^{\infty}\sum_{k=0}^{\infty} \frac{(A)_{n+k} \ \mathrm{u}^{\mathrm{n}} v^k}{n!\, k!}, \quad \left(|u + v| < 1\right). \tag{9}$$

If $A(k, n)$ and $B(k, n)$ are matrices in $C^{N\times N}$ for $n \ge 0, k \ge 0,$ we have the following relations $[2]$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k) \tag{10}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k) \tag{11}$$

similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k) \tag{12}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}B(k,n-k) \tag{13}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k). \tag{14}$$

If A is a positive stable matrix in $C^{N\times N}$ and satisfying the condition $(-z/2)\notin\sigma(A)\,\forall z\in Z^{+}\cup\{0\}$ then the Gegenbauer matrix polynomials of one variable is defined by $[10]$

$$C_n^A(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k (A)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}.$$

If A is a positive stable matrix in $C^{N\times N}$ then the Gegenbauer matrix polynomials of one variable is defined by $[1]$

$$C_n^{\nu}(x,A)=\sum_{k=0}^{[n/2]}\frac{(-1)^k (\nu)_{n-k}}{k!(n-2k)!}\left(\sqrt{2A}\,x\right)^{n-2k}, \tag{15}$$

and the generating function for these polynomials is

$$\left(I-xt\sqrt{2A}+t^2 I\right)^{-\nu}=\sum_{n=0}^{\infty}C_n^{\nu}(x,A)t^n.$$

The two- variable Hermite matrix polynomials define by $[19]$

$$H_n(x,y,A)=n!\sum_{n=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(x\sqrt{2A}\right)^{n-2k}y^k}{k!(n-2k)!}, \tag{16}$$

and

$$x^n\left(\sqrt{2A}\right)^n=\sum_{k=0}^{[n/2]}\frac{n!y^k}{(n-2k)!k!}H_{n-2k}(x,y,A)\ . \tag{17}$$

The Gegenbauer matrix polynomials of two variables by the double generating relation define by [18]

$$\left(1-2xs+s^2-2yt+t^2\right)^{-A}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{A}x\ (x,y)s^n t^k, \tag{18}$$

where A be a positive stable matrix in $C^{N\times N}$. From (18); it follows that

$$C^{A}_{n,k}(\mathrm{x},\mathrm{y})=\sum_{\mathrm{r}=0}^{\left[\frac{\mathrm{n}}{2}\right]}\sum_{\mathrm{j}=0}^{\left[\frac{\mathrm{k}}{2}\right]}\frac{(-1)^{r+j}\ (A)_{n+k-r-j}\ (2\mathrm{x})^{n-2r}\ (2\mathrm{y})^{k-2j}}{\mathrm{r}!\ \mathrm{j}!\ (\mathrm{n}-2\mathrm{r})!\ (\mathrm{k}-2\mathrm{j})!}, \tag{19}$$

where $C_{n,k}^{A}$ (x, y) is a matrix polynomial in two variables x and y of degree precisely n in x and k in y.

The objective of this paper is to drive and study of some properties of $C_{n,k}^{\nu}$ (x, y,A), matrix differential recurrence relations, partial matrix differential equation, triple Hypergeometric matrix forms and a bilinear double generating matrix function and a special property. Finally, we study and expand the Gegenbauer matrix polynomials of two variables in series of two variable of Hermite matrix polynomials for the new kind of Gegenbauer matrix of two variable polynomials.

2. The Gegenbauer matrix polynomials of two variables: Let A be a positive stable matrix in $C^{N\times N}$ we define the Gegenbauer matrix polynomials of two variables $C_{n,k}^{\nu}$ (x, y,A) by the double generating relation

$$\left(I-\sqrt{2A}\,xs+s^2I-\sqrt{2A}\,yt+t^2I\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{\nu}_{n,k}\ (\mathrm{x},\mathrm{y},\mathrm{A})\mathrm{s}^n t^k\quad (\nu\neq 0), \tag{20}$$

where $\left\|\sqrt{2A}\,xs-s^2I+\sqrt{2A}\,yt-t^2I\right\|<1$.

In which $\left(I-\sqrt{2A}\,xs+s^2I-\sqrt{2A}\,yt+t^2I\right)^{-\nu}$ denotes the particular branch which $\to I$, where I is the identity matrix in $C^{N\times N}$, as $s\to 0$ and $t\to 0$. We shall first show that $C^{\nu}_{n,k}$ (x, y, A) is a polynomial of degree precisely n in x and k in y.

Relation (9) enables us to rewrite r. h. s. of (20) as

$$\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\nu)_{n+k}\,\left(\sqrt{2A}\mathrm{xs}-\mathrm{s}^2\right)^{\mathrm{n}}\,\left(\sqrt{2A}\mathrm{yt}-\mathrm{t}^2\right)^{\mathrm{k}}}{n!\ \mathrm{k}!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{k}\frac{(\nu)_{n+k}\,\left(\sqrt{2A}\mathrm{xs}\right)^{\mathrm{n}}\,\left(\sqrt{2A}\mathrm{yt}\right)^{\mathrm{k}}\,(-\mathrm{n})_{\mathrm{r}}\,(-k)_{\mathrm{j}}}{n!\ \mathrm{k}!\ \mathrm{r}!\ \mathrm{j}!}\left(\frac{s}{\sqrt{2A}x}\right)^{r}\left(\frac{t}{\sqrt{2A}y}\right)^{j}$$

Using (13), we obtain

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(\nu)_{n+k-r-j}\,\left(\sqrt{2A}\,\mathrm{x}\right)^{\mathrm{n-2r}}\left(\sqrt{2A}\,\mathrm{y}\right)^{\mathrm{k-2j}}(-1)^{\mathrm{r+j}}\ \mathrm{s}^{\mathrm{n}}\ \mathrm{t}^{\mathrm{k}}}{\mathrm{r}!\ \mathrm{j}!(\mathrm{n}-2\mathrm{r})!\ (\mathrm{k}-2\mathrm{j})!}.$$

Thus, from (20) and above relation by comparing the coefficient of $s^n\,t^k$,we obtain the following explicit representation for the Gegenbauer matrix polynomials of two variables $C^{\nu}_{n,k}$(x, y, A)

$$C^{\nu}_{n,k}\left(\mathrm{x},\mathrm{y},\mathrm{A}\right)=\sum_{\mathrm{r}=0}^{\left[\frac{\mathrm{n}}{2}\right]}\sum_{\mathrm{j}=0}^{\left[\frac{\mathrm{k}}{2}\right]}\frac{(-1)^{r+j}\,(\nu)_{n+k-r-j}\left(\sqrt{2\mathrm{A}}\mathrm{x}\right)^{n-2r}\left(\sqrt{2\mathrm{A}}\mathrm{y}\right)^{k-2j}}{\mathrm{r}!\ \mathrm{j}!\ (\mathrm{n}-2\mathrm{r})!\ (\mathrm{k}-2\mathrm{j})!}. \qquad (21)$$

Clearly, $C^{\nu}_{n,k}$(x, y, A) is a polynomial in two variables x and y of degree precisely n in x and k in y. Thus $C^{\nu}_{n,k}$(x, y, A) is a polynomial in two variables x and y of degree $n+k$. From equation (21) it is clear that

$$C_{n,k}^{\nu}\left(\mathrm{x},\mathrm{y},\mathrm{A}\right)=\frac{2^{n+k}\left(\nu\right)_{n+k}\ \mathrm{x}^{n}\ \mathrm{y}^{k}}{\mathrm{n}!\ \mathrm{k}!}+\pi, \qquad (22)$$

where π is a polynomial in two variables x and y of degree $n+k-2$.

It is indeed easy to note the following some special cases:
If in (20) we replace x by $-x$ and s by $-s$, the left member does not change. Hence

$$C_{n,k}^{\nu}\left(-\mathrm{x},\mathrm{y},\mathrm{A}\right)=\left(-1\right)^{n}C_{n,k}^{\nu}\left(\mathrm{x},\mathrm{y},\mathrm{A}\right). \qquad (23)$$

Again, by replacing y by $-y$ and t by $-t$ in (20), we find

$$C_{n,k}^{\nu}\left(\mathrm{x},-\mathrm{y},\mathrm{A}\right)=\left(-1\right)^{k}C_{n,k}^{\nu}\left(\mathrm{x},\mathrm{y},\mathrm{A}\right). \qquad (24)$$

Clearly, $C_{n,k}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A})$ is an odd function of x for n odd, an even function of x for n even. Similarly $C_{n,k}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A})$ is an odd function of y for k odd, an even function of y for k even.

Replacing x by $-x$, y by $-y$, s by $-s$ and t by $-t$ in (20), we obtain

$$C_{n,k}^{\nu}(-\mathrm{x},-\mathrm{y},\mathrm{A})=\left(-1\right)^{n+k}C_{n,k}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A}). \qquad (25)$$

For $t=0$ in (20), it follows

$$C_{n,0}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A})=C_{n}^{\nu}(\mathrm{x},\mathrm{A}), \qquad (26)$$

Where $C_{n}^{\nu}(\mathrm{x},\mathrm{A})$ is the well-known Gegenbauer matrix polynomial (15). Taking $s=0$ in Eq. (20), we obtain

$$C_{0,k}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A})\ =\ C_{k}^{\nu}(\mathrm{y},\mathrm{A}). \qquad (27)$$

Further taking $x=0$ and $y=0$ in (20), we find

$$\left(I+s^{2}I+t^{2}I\right)^{-\nu}=\sum_{\mathrm{n}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty}C_{n,k}^{\nu}\left(0,0,\mathrm{A}\right)\mathrm{s}^{\mathrm{n}}\mathrm{t}^{\mathrm{k}}. \qquad (28)$$

But, in view of the relation (9) it follows that

$$\left(I+s^{2}I+t^{2}I\right)^{-\nu}=\sum_{\mathrm{n}=0}^{\infty}\sum_{\mathrm{k}=0}^{\infty}\frac{\left(-1\right)^{\mathrm{n+k}}\left(\nu\right)_{\mathrm{n+k}}\ \mathrm{s}^{2\mathrm{n}}\ \mathrm{t}^{2\mathrm{k}}}{n!\ \mathrm{k}!}.$$

Hence

$$\left.\begin{array}{l} C^{\nu}_{2n+1,2k}\left(0,\,0,\mathrm{A}\right)=0,\;C^{\nu}_{2n,2k+1}\left(0,\,0,\mathrm{A}\right)=0,\;C^{\nu}_{2n+1,2k+1}\left(0,\,0,\mathrm{A}\right)=0,\\[2ex] C^{\nu}_{2n,2k}\left(0,\,0,\mathrm{A}\right)=\dfrac{\left(-1\right)^{\mathrm{n+k}}\left(\nu\right)_{\mathrm{n+k}}}{n!\;\mathrm{k}!}\end{array}\right\}. \tag{29}$$

It is clear from (21) that

$$\frac{\partial}{\partial x}\,C^{\nu}_{n,k}\left(\mathrm{x,\,y,A}\right)=\sum_{\mathrm{r=0}}^{\left[\frac{\mathrm{n-1}}{2}\right]}\sum_{\mathrm{j=0}}^{\left[\frac{\mathrm{k}}{2}\right]}\frac{\left(-1\right)^{r+j}\left(\nu\right)_{n+k-r-j}\sqrt{2\mathrm{A}}\left(\sqrt{2\mathrm{A}}\mathrm{x}\right)^{n-1-2r}\left(\sqrt{2\mathrm{A}}\mathrm{y}\right)^{k-2j}}{\mathrm{r!\;\,j!(n-1-2r)!(k-2j)!}}. \tag{30}$$

Therefore, we have

$$\left[\frac{\partial}{\partial x}\,C^{\nu}_{2n+1,2k}\left(\mathrm{x,\,y,A}\right)\right]_{x=0,\,y=0}=\frac{\left(-1\right)^{n+k}\sqrt{2\mathrm{A}}\left(\nu\right)_{n+k+1}}{\mathrm{n!\;\,k!}}, \tag{31}$$

$$\left.\begin{array}{lcl} \left[\dfrac{\partial}{\partial x}\,C^{\nu}_{2n,2k}=\left(\mathrm{x,\,y,A}\right)\right]_{x=0,\,y=0} & = & 0,\\[2ex] \left[\dfrac{\partial}{\partial x}\,C^{\nu}_{2n,2k+1}\left(\mathrm{x,\,y,A}\right)\right]_{x=0,\,y=0} & = & 0\\[2ex] \left[\dfrac{\partial}{\partial x}\,C^{\nu}_{2n+1,2k+1}\left(\mathrm{x,\,y,A}\right)\right]_{x=0,\,y=0} & = & 0,\end{array}\right\}, \tag{32}$$

and

$$\frac{\partial}{\partial y}\,C^{\nu}_{n,k}\left(\mathrm{x,\,y,A}\right)=\sum_{\mathrm{r=0}}^{\left[\frac{\mathrm{n}}{2}\right]}\sum_{\mathrm{j=0}}^{\left[\frac{\mathrm{k-1}}{2}\right]}\frac{\left(-1\right)^{r+j}\left(\nu\right)_{n+k-r-j}\left(\sqrt{2\mathrm{A}}\mathrm{x}\right)^{n-2r}\sqrt{2\mathrm{A}}\left(\sqrt{2\mathrm{A}}\mathrm{y}\right)^{k-1-2j}}{\mathrm{r!\;\,j!(n-2r)!\;(k-1-2j)!}}. \tag{33}$$

Similarly, we have

$$\left[\frac{\partial}{\partial y}\, C^{\nu}_{2n,2k+1}(\mathrm{x},\mathrm{y},\mathrm{A})\right]_{x=0,y=0} = \frac{(-1)^{n+k}\sqrt{2\,\mathrm{A}}\,(\nu)_{n+k+1}}{\mathrm{n}\,!\;\mathrm{k}\,!}, \tag{34}$$

$$\left.\begin{aligned}\left[\frac{\partial}{\partial y}\, C^{\nu}_{2n,2k}(\mathrm{x},\mathrm{y},\mathrm{A})\right]_{x=0,y=0} &= 0,\\ \left[\frac{\partial}{\partial y}\, C^{\nu}_{2n+1,2k}(\mathrm{x},\mathrm{y},\mathrm{A})\right]_{x=0,y=0} &= 0,\\ \left[\frac{\partial}{\partial y}\, C^{\nu}_{2n+1,2k+1}(\mathrm{x},\mathrm{y},\mathrm{A})\right]_{x=0,y=0} &= 0,\end{aligned}\right\}. \tag{35}$$

Remarks:

2.1. For $\nu=-1/2$ and $t=0$, and using (8) and (13), the equation (20) reduce to the Legendre matrix polynomials given by [20]

$$P_m(\mathrm{x},\mathrm{A})=\sum_{\mathrm{j}=0}^{\left[\frac{\mathrm{m}}{2}\right]}\frac{(-1)^j\,(1/2)_{m-j}\left(\sqrt{2\mathrm{A}}\mathrm{x}\right)^{m-2j}}{\mathrm{j}!\,(m-2\mathrm{j})!}. \tag{36}$$

2.2. Taking $\nu=-1$ and $t=0$ in equation (20) and using (8) and (13), we get the Chebyshev matrix polynomial of second kind [12]

$$U_n(\mathrm{x},\mathrm{A})=\sum_{\mathrm{j}=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j\,(n-j)!\left(\sqrt{2\mathrm{A}}\mathrm{x}\right)^{n-2j}}{\mathrm{j}!\,(n-2\mathrm{j})!}. \tag{37}$$

3. Matrix differential recurrence relations: The following recurrence relations hold for Gegenbauer matrix polynomials of two variables

I. $$\sqrt{2A}x\frac{\partial}{\partial x} C^{\nu}_{n,k}\left(\text{x, y,A}\right)-2I\frac{\partial}{\partial x}C^{\nu}_{n-1,k}\left(\text{x, y,A}\right)=\sqrt{2A}\; n\, C^{\nu}_{n,k}\left(\text{x, y,A}\right) \qquad (38)$$

II. $$\sqrt{2A}y\frac{\partial}{\partial y} C^{\nu}_{n,k}\left(\text{x, y,A}\right)-2I\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\text{x, y,A}\right)=\sqrt{2A}\; k\, C^{\nu}_{n,k}\left(\text{x, y,A}\right) \qquad (39)$$

III. $$\sqrt{2A}\,(\nu+k+n)C^{\nu}_{n,k}\left(\text{x, y,A}\right)$$
$$=I\frac{\partial}{\partial x}C^{\nu}_{n+1,k}\left(\text{x, y,A}\right)-I\frac{\partial}{\partial x}C^{\nu}_{n-1,k}\left(\text{x, y,A}\right)-I\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\text{x, y,A}\right) \qquad (40)$$

IV. $$\left(\left(\sqrt{2A}x\right)^2-4I\right)\frac{\partial}{\partial x}C^{\nu}_{n,k}\left(\text{x,y,A}\right)$$

$$=n\left(\sqrt{2A}\right)^2 x\,C^{\nu}_{n,k}\left(\text{x, y,A}\right)-4I\frac{\partial}{\partial y}C^{\nu}_{n-1,k-1}\left(\text{x,y,A}\right)-2\sqrt{2A}\,I\left(n+2k+2\nu-1\right)C^{\nu}_{n-1,k}\left(\text{x,y,A}\right). \qquad (41)$$

V. $$\left(\left(\sqrt{2A}\,y\right)^2-4I\right)\frac{\partial}{\partial y}C^{\nu}_{n,k}\left(\text{x,y,A}\right)$$

$$=\left(\sqrt{2A}\right)^2 k\,y\,C^{\nu}_{n,k}\left(\text{x, y,A}\right)-4I\frac{\partial}{\partial x}C^{\nu}_{n-1,k-1}\left(\text{x,y,A}\right)-2\sqrt{2A}\,I\left(2n+k+2\nu-1\right)C^{\nu}_{n,k-1}\left(\text{x,y,A}\right). \qquad (42)$$

Proof of (38)**.** Let, $G=\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu}$ (43)

differentiating (43) with respect to x and s yields respectively

$$\frac{\partial G}{\partial x}=\frac{\sqrt{2A}\,\text{s}\nu G}{\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)}, \qquad (44)$$

and

$$\frac{\partial G}{\partial s}=\frac{\left(\sqrt{2A}\,\text{x}-2sI\right)\nu G}{\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)}. \qquad (45)$$

So that matrix function G satisfies the partial matrix differential equation

$$\left(\sqrt{2A}\,x\ -2sI\right)\frac{\partial G}{\partial x}\ -\sqrt{2A}\,s\,\frac{\partial G}{\partial s}\ =0.$$

Therefore, by using (20) and (43)

$$\sqrt{2A}\,s\,n\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}\ (\mathrm{x},\mathrm{y},\mathrm{A})s^{n-1}t^{k}$$
$$=\sqrt{2A}\,x\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n,k}^{\nu}\,(\mathrm{x},\mathrm{y},\mathrm{A})s^{n}t^{k}-2I\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C_{n-1,k}^{\nu}\ (\mathrm{x},\mathrm{y},\mathrm{A})s^{n}t^{k}.$$

Since $\frac{\partial}{\partial x}C_{0,k}^{\nu}\,(\mathrm{x},\mathrm{y},\mathrm{A})=0,\ k\geq 0,$ and for $n\geq 1$, then by identification of the coefficients of $s^{n}t^{k}$, (38) is proved.

Proof of (39)**.** Similarly differentiating (43) with respect to y and t yields respectively

$$\frac{\partial G}{\partial y}=\frac{\sqrt{2A}\ \mathrm{t}\,\nu G}{\left(I-\sqrt{2A}xs+s^{2}I-\sqrt{2A}yt+t^{2}I\right)}, \tag{46}$$

and

$$\frac{\partial G}{\partial t}=\frac{\left(\sqrt{2A}\ \mathrm{x}-2tI\right)\nu G}{\left(I-\sqrt{2A}xs+s^{2}I-\sqrt{2A}yt+t^{2}I\right)}. \tag{47}$$

So that matrix function G satisfies the partial matrix differential equation

$$\left(\sqrt{2A}\,y\ -2t\,I\right)\frac{\partial G}{\partial y}\ -\sqrt{2A}\,t\,\frac{\partial G}{\partial t}\ =0.$$

Therefore, by using (20) and (43)

$$\sqrt{2A}\,t\,k\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}\,(\mathrm{x},\mathrm{y},\mathrm{A})s^{n}\,t^{k-1}$$
$$=\sqrt{2A}\,y\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k}^{\nu}\,(\mathrm{x},\mathrm{y},\mathrm{A})s^{n}t^{k}-2I\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C_{n,k-1}^{\nu}(\mathrm{x},\mathrm{y},\mathrm{A})s^{n}t^{k}.$$

Since $\frac{\partial}{\partial y} C^{\nu}_{n,0}\left(\mathrm{x, y, A}\right)=0, n \ge 0,$ and for $k \ge 1$, then by identification of the coefficients of $s^n t^k$, (39) is proved.

Now, using (38) and (39), we get

$$\sqrt{2A}\,\left(n+k\right) C^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)$$
$$=\sqrt{2A}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)-2I\left(\frac{\partial}{\partial x}C^{\nu}_{n-1,k}\left(\mathrm{x, y, A}\right)+\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\mathrm{x, y, A}\right)\right). \qquad (48)$$

Proof of (40)**.** Now from (43) and (44), (45) with the aid of (20) we get respectively the following

$$\sqrt{2A}\,\nu\,\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu-1}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}C^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)\,s^{n-1}t^k, \qquad (49)$$

and

$$\left(\sqrt{2A}\,x-2sI\right)\nu\,\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu-1}=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}nC^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)\,s^{n-1}t^k. \qquad (50)$$

Again from (43) and (46), (47) with the aid of (20) we get respectively the following

$$\sqrt{2A}\,\nu\,\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}C^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)\,s^{n}t^{k-1}, \qquad (51)$$

and

$$\left(\sqrt{2A}\,y-2tI\right)\nu\,\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}k\,C^{\nu}_{n,k}\left(\mathrm{x, y, A}\right)\,s^{n}t^{k-1}. \qquad (52)$$

Since

$$I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I=I-s^2I-t^2I-s\left(\sqrt{2A}x-2sI\right)-t\left(\sqrt{2A}y-2tI\right).$$

If we multiply the equation (49) by $\frac{I-s^2I}{\sqrt{2A}}$ and (50) by s and subtracting (50) from (49) we have

$$\nu\left(I-\sqrt{2A}\,xs+s^2I\right)\,\left(I-\sqrt{2A}s+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu-1}$$

$$=\frac{I-s^2I}{\sqrt{2A}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial x}k\,C_{n,k}^{\nu}\left(\text{x, y,A}\right)\,s^{n-1}t^{k}\,-n\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\,C_{n,k}^{\nu}\left(\text{x, y,A}\right)\,s^{n}t^{k}\,. \tag{53}$$

And by multiplying the equation (51) by $\frac{-t^2I}{\sqrt{2A}}$ and (52) by t and subtracting (52) from (51) we get

$$\nu\left(\sqrt{2A}\,y\,t-t^2I\right)\,\left(I-\sqrt{2A}s+s^2I-\sqrt{2A}y\,t+t^2I\right)^{-\nu-1}$$

$$=\frac{t^2I}{\sqrt{2A}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\partial}{\partial y}k\,C_{n,k}^{\nu}\left(\text{x, y,A}\right)\,s^{n}t^{k-1}+k\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\,C_{n,k}^{\nu}\left(\text{x, y,A}\right)\,s^{n}t^{k}\,. \tag{54}$$

Now subtracting (54) from (53), and then equating for $S^n\,t^k$ thus the matrix differential recurrence relation (40) is established.

Similarly, we can get another matrix differential recurrence relation on similar lines in the form,

$$\sqrt{2A}(\nu+k+n)C_{n,k}^{\nu}\left(\text{x, y,A}\right)$$

$$=I\,\frac{\partial}{\partial y}C_{n,k+1}^{\nu}\left(\text{x, y,A}\right)-I\,\frac{\partial}{\partial y}C_{n,k-1}^{\nu}\left(\text{x, y,A}\right)-I\,\frac{\partial}{\partial x}C_{n-1,k}^{\nu}\left(\text{x, y,A}\right). \tag{55}$$

Proof of (41)**.** Adding (40) successively to (38), we obtain

$$\sqrt{2A}\,x\frac{\partial}{\partial x}C_{n,k}^{\nu}\left(\text{x, y,A}\right)$$

$$=2I\,\frac{\partial}{\partial x}C_{n+1,k}^{\nu}\left(\text{x, y, A}\right)-2I\,\frac{\partial}{\partial y}C_{n,k-1}^{\nu}\left(\text{x, y, A}\right)-\sqrt{2A}\left(n+2k+2\nu\right)C_{n,k}^{\nu}\left(\text{x, y, A}\right). \tag{56}$$

Replacing n by $n-1$ in (56) and using (38), thus the relation (41) is established.

Proof of (42)**.** Similarly if we adding (55) successively to (39), we obtain

$$\sqrt{2A}\,y\,\frac{\partial}{\partial y}C_{n,k}^{\nu}\left(\text{x, y,A}\right)$$

$$=2I\,\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(\mathrm{x,\,y,A})-2I\,\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(\mathrm{x,\,y,A})-\sqrt{2A}\,(2n+k+2\nu)C^{\nu}_{n,k}(\mathrm{x,\,y,A}). \qquad (57)$$

Replacing k by $k-1$ in (57) and using (39) ,thus the relation (42) is established.

From (41) and (42), we get the recurrence relation

$$\left(\left[\left(\sqrt{2A}x\right)^2-4I\right]\frac{\partial}{\partial x}+\left[\left(\sqrt{2A}y\right)^2-4I\right]\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(\mathrm{x,y,A})$$

$$=-2\sqrt{2A}\,I\,(n+2k+2\nu-1)C^{\nu}_{n-1,k}(\mathrm{x,y,A})-2\sqrt{2A}\,I\,(2n+k+2\nu-1)C^{\nu}_{n,k-1}(\mathrm{x,y,A})$$

$$-4I\left\{\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\right\}C^{\nu}_{n-1,k-1}(\mathrm{x,\,y,A})+\left(\sqrt{2A}\right)^2(n\,x+k\,y)C^{\nu}_{n,k}(\mathrm{x,\,y,A}). \qquad (58)$$

Also, it is easy to drive the following matrix deferential recurrence relation

$$\sqrt{2A}\,y\,\frac{\partial}{\partial y}C^{\nu}_{n,k}(\mathrm{x,\,y,A})$$

$$=2I\,\frac{\partial}{\partial x}C^{\nu}_{n+1,k}(\mathrm{x,\,y,A})-2I\,\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(\mathrm{x,\,y,A})-\sqrt{2A}\,(2n+k+2\nu)C^{\nu}_{n,k}(\mathrm{x,\,y,A}), \qquad (59)$$

and

$$\sqrt{2A}\left(x\,\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(\mathrm{x,\,y,A})=2I\,\frac{\partial}{\partial x}C^{\nu}_{n+1,k}(\mathrm{x,\,y,A})-\sqrt{2A}\,(n+k+2\nu)C^{\nu}_{n,k}(\mathrm{x,\,y,A}). \qquad (60)$$

Proof. By adding (40) successively to (39) and (48), then (59) and (60) can be proved respectively.

Also, we can drive the following matrix deferential recurrence relation

$$\sqrt{2A}\,x\,\frac{\partial}{\partial x}C^{\nu}_{n,k}(\mathrm{x,\,y,A})$$

$$=2I\,\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(\mathrm{x,\,y,A})-2I\,\frac{\partial}{\partial y}C^{\nu}_{n,k-1}(\mathrm{x,\,y,A})-\sqrt{2A}\,(n+2k+2\nu)C^{\nu}_{n,k}(\mathrm{x,\,y,A}), \qquad (61)$$

and

$$\sqrt{2A}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)C^{\nu}_{n,k}(\text{x, y,A})=2I\frac{\partial}{\partial y}C^{\nu}_{n,k+1}(\text{x,y,A})-\sqrt{2A}(n+k+2\nu)C^{\nu}_{n,k}(\text{x,y,A}). \quad (62)$$

Proof. By adding (55) successively to (38) and (48), then (61) and (62) can be proved respectively.

4. Partial matrix differential equation of $C^{\nu}_{n,k}(\text{x, y, A})$: Partial matrix differential equation of Gegenbauer matrix polynomials of two variables are given as follows

$$\left\{\left(4I-\sqrt{2A}\,x^2\right)\frac{\partial^2}{\partial x^2}-\left(4I-\sqrt{2A}\,y^2\right)\frac{\partial^2}{\partial y^2}\right\}C^{\nu}_{n,k}(\text{x,y,A})$$

$$-\left(\sqrt{2A}\right)^2\left\{(2k+2\nu+1)x\frac{\partial}{\partial x}-(2n+2\nu+1)y\frac{\partial}{\partial y}\right\}C^{\nu}_{n,k}(\text{x,y,A})+\left(\sqrt{2A}\right)^2(n-k)(n+k+2\nu)C^{\nu}_{n,k}(\text{x, y,A})=0 \quad (63)$$

Proof of (63). Differentiation (38), partially with respect to x, we get

$$2I\frac{\partial^2}{\partial x^2}C^{\nu}_{n-1,k}(\text{x, y,A})=\sqrt{2A}\,x\frac{\partial^2}{\partial x^2}C^{\nu}_{n,k}(\text{x, y,A})+\sqrt{2A}(1-n)\frac{\partial}{\partial x}C^{\nu}_{n,k}(\text{x, y,A}). \quad (64)$$

Now by shifting the index from n to $n-1$ in (56), we get

$$\sqrt{2A}\,x\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(\text{x, y,A})$$

$$=2I\frac{\partial}{\partial x}C^{\nu}_{n,k}(\text{x, y,A})-2I\frac{\partial}{\partial y}C^{\nu}_{n-1,k-1}(\text{x, y,A})-\sqrt{2A}(n+2k+2\nu-1)C^{\nu}_{n-1,k}(\text{x, y,A}), \quad (65)$$

differentiate (65) with respect to x, we get

$$\sqrt{2A}\,x\frac{\partial^2}{\partial x^2}C^{\nu}_{n-1,k}(\text{x,y,A})$$

$$=2I\frac{\partial^2}{\partial x^2}C^{\nu}_{n,k}(\text{x,y,A})-\sqrt{2A}(n+2k+2\nu)\frac{\partial}{\partial x}C^{\nu}_{n-1,k}(\text{x,y,A})-2I\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}(\text{x,y,A}). \quad (66)$$

From (38) and (64) by substituting $\frac{\partial}{\partial x}C^{\nu}_{n-1,k}\left(\mathrm{x,y,A}\right)$ and $\frac{\partial^2}{\partial x^2}C^{\nu}_{n-1,k}\left(\mathrm{x,y,A}\right)$ into (66) and rearrangement terms yield

$$\left(4I-\left(\sqrt{2A}x\right)^2\right)\frac{\partial^2}{\partial x^2}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)-\left(\sqrt{2A}\right)^2\left(2k+2\nu+1\right)x\frac{\partial}{\partial x}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)$$
$$+\left(\sqrt{2A}\right)^2 n\left(n+2k+2\nu\right)C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)-4I\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}\left(\mathrm{x,y,A}\right)=0. \qquad (67)$$

Similarly, by differentiate (39) with respect to y, we get

$$2I\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k-1}\left(\mathrm{x,\,y,A}\right)=\sqrt{2A}\,y\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k}\left(\mathrm{x,\,y,A}\right)+\sqrt{2A}\left(1-k\right)\frac{\partial}{\partial y}C^{\nu}_{n,k}\left(\mathrm{x,\,y,A}\right). \qquad (68)$$

If we replace k with $k-1$ in (57), we get

$$\sqrt{2A}y\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)$$
$$=2I\frac{\partial}{\partial y}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)-\sqrt{2A}\left(2n+k+2\nu-1\right)C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)-2I\frac{\partial}{\partial x}C^{\nu}_{n-1,k-1}\left(\mathrm{x,y,A}\right), \qquad (69)$$

differentiation (69) partially with respect to y, we get

$$\sqrt{2A}y\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)$$
$$=2I\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)-\sqrt{2A}\left(2n+k+2\nu\right)\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)-2I\frac{\partial^2}{\partial x\,\partial y}C^{\nu}_{n-1,k-1}\left(\mathrm{x,y,A}\right). \qquad (70)$$

From (39) and (68) by substituting $\frac{\partial}{\partial y}C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)$ and $\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k-1}\left(\mathrm{x,y,A}\right)$ into (70) and rearrangement terms yield

$$\left(4I-\left(\sqrt{2A}\,y\right)^2\right)\frac{\partial^2}{\partial y^2}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)-\left(\sqrt{2A}\right)^2\left(2k+2\nu+1\right)y\frac{\partial}{\partial y}C^{\nu}_{n,k}\left(\mathrm{x,y,A}\right)$$

$$+\left(\sqrt{2A}\right)^2 k\left(2n+k+2\nu\right)C_{n,k}^{\nu}\left(\mathrm{x,y,A}\right)-4I\,\frac{\partial^2}{\partial x\,\partial y}C_{n-1,k-1}^{\nu}\left(\mathrm{x,y,A}\right)=0, \qquad (71)$$

subtracting (71) from (67), thus the relation (63) is established.

5. Triple hypergeometric matrix forms of $C_{n,k}^{\nu}(\mathrm{x,\,y,A})$: For the Gegenbauer matrix polynomials of two variables, which is given by (20), we can easily derive the following hypergeometric matrix for the $C_{n,k}^{\nu}(\mathrm{x,\,y,A})$

$$C_{n,k}^{\nu}\left(x,y,A\right)=\frac{\left(2\nu\right)_{\mathrm{n+k}}}{\mathrm{n!\;k!}}\,\mathrm{F}^{(3)}\left[\begin{matrix}-::-nI;-kI;2\nu+nI+kI:-;-;-; \\ \qquad\qquad\qquad\qquad \frac{2I-\sqrt{2A}\,x}{4},\ \frac{2I-\sqrt{2A}\,y}{4},\frac{1}{2} \\ \frac{2\nu+I}{2}::-;-;-:-;-;-;\end{matrix}\right]. \qquad (72)$$

$$C_{n,k}^{\nu}(x,y,A)=\frac{\left(\sqrt{2A}\right)^{n+k}\left(\nu\right)_{n+k}\,\mathrm{x^n}y^{k}}{\mathrm{n!\;k!}}\,\mathrm{F}\left[\begin{matrix}-:-\frac{n}{2}I,-\frac{n}{2}I+\frac{I}{2};-\frac{k}{2}I,-\frac{k}{2}I+\frac{I}{2}\ ; \\ \qquad\qquad\qquad\qquad \frac{1}{x^2},\ \frac{1}{y^2} \\ I-\nu-\mathrm{nI}-\mathrm{kI}\ :\ -;\ -\ ;\end{matrix}\right]. \qquad (73)$$

Proof of (72). Now by taking left hand side of (20)

$$\left(I-\sqrt{2A}xs+s^2I-\sqrt{2A}yt+t^2I\right)^{-\nu}=\left[\left(I-s-t\right)^2-s\left(\sqrt{2A}\,x-2I\right)-t\left(\sqrt{2A}\,y-2I\right)-2st\right]^{-\nu}$$

$$=\left(I-s-t\right)^{-2\nu}\left[1-\frac{s\left(\sqrt{2A}\,x-2I\right)}{\left(I-s-t\right)^2}-\frac{t\left(\sqrt{2A}\,y-2I\right)}{\left(I-s-t\right)^2}-\frac{2st}{\left(I-s-t\right)^2}\right]^{-\nu},$$

now by using (9)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}(x,y,A)s^n t^k = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{2^r(\nu)_{j+p+r}\left(\sqrt{2A}\,x-2I\right)^j\left(\sqrt{2A}\,y-2I\right)^p s^{j+r}t^{p+r}}{j\,!\,p\,!\,r\,!\,(I-s-t)^{2j+2p+2r+2\nu}}$$

$$= \sum_{n,k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{2^r(\nu)_{j+p+r}\left(\sqrt{2A}\,x-2I\right)^j\left(\sqrt{2A}\,y-2I\right)^p (2\nu)_{2j+2p+2r+n+k}\, s^{n+j+r}t^{k+p+r}}{j\,!\,p\,!\,r\,!\,n\,!\,k\,!(2\nu)_{2j+2p+2r}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)} \frac{2^r(\nu)_{j+p+r}\left(\sqrt{2A}\,x-2I\right)^j\left(\sqrt{2A}\,y-2I\right)^p (2\nu)_{n+k+j+p}\, s^n t^k}{j\,!p!\,r!\,(n-j-r)!(k-p-r)!\,(2\nu)_{2j+2p+2r}}$$

$$= \sum_{n,k=0}^{\infty}\sum_{j=0}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)} \frac{(2\nu)_{n+k+j+p}\left(\sqrt{2A}\,x-2I\right)^j\left(\sqrt{2A}\,y-2I\right)^p s^n t^k}{2^{2j+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!(n-j-r)!\,(k-p-r)!}$$

$$= \sum_{n,k=0}^{\infty}\sum_{j=0}^{n}\frac{(2\nu)_{n+k}}{n!k!}$$

$$\times\sum_{j}^{n}\sum_{p=0}^{k}\sum_{r=0}^{\min(n,k)} \frac{(2\nu+n+k)_{j+p}(-n)_{j+r}(-k)_{p+r}\left(\left(2I-\sqrt{2A}\,x\right)/4\right)^j\left(\left(2I-\sqrt{2A}\,y\right)/4\right)^p s^n t^k}{2^r\left(\frac{2\nu+1}{2}\right)_{j+p+r} j\,!p!\,r!\,n!\,k!}.$$

Comparing the coefficient of $s^n\, t^k$ from both sides of the above, we get (72).

Using (25) together with (72) , it follows that

$$C_{n,k}^{\nu}(x,y,A)=\frac{(-1)^{n+k}(2\nu)_{n+k}}{n\,!\,k\,!}\mathrm{F}^{(3)}\left[\begin{matrix}-::-nI;-kI;2\nu+nI+kI:-;-;-; \\ \\ \frac{2\nu+I}{2}::\ -;-;-:-;-;-;\end{matrix}\ \frac{2I+\sqrt{2A}\,x}{4},\ \frac{2I+\sqrt{2A}\,y}{4},\frac{1}{2}\right]. \tag{74}$$

Proof of (73)**.** Next, consider (21) again

$$C_{n,k}^{\nu}(x,y,A)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^{r+j}(\nu)_{n+k-r-j}\left(\sqrt{2A}\,x\right)^{n-2r}\left(\sqrt{2A}\,y\right)^{k-2j}}{r!\,j!\,(n-2r)!\,(k-2j)!},$$

we may write it as

$$C_{n,k}^{\nu}(x,y,A)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{(-n)_{2r}(-k)_{2j}(\nu)_{n+k}\left(\sqrt{2A}\,x\right)^{n-2r}\left(\sqrt{2A}\,y\right)^{k-2j}}{r!\,j!\,n!\,k!\,(1-\nu-n-k)_{r+j}}$$

$$=\frac{\left(\sqrt{2A}\right)^{n+k}(\nu)_{n+k}\,x^{n}y^{k}}{n!\,k!}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{\left(-\frac{n}{2}\right)_{r}\left(-\frac{n}{2}+\frac{1}{2}\right)_{r}\left(-\frac{k}{2}\right)_{j}\left(-\frac{k}{2}+\frac{1}{2}\right)_{j}}{r!\,j!\,(1-\nu-n-k)_{r+j}\,x^{2r}y^{2j}}.$$

Which can be written in terms of Kampé de Fériet's double hypergeometric matrix function, thus the proof of (73) is completed.

6. Double generating matrix functions and properties of $C_{n,k}^{\nu}(x,y,A)$: The generating matrix function $\left(I-\sqrt{2A}\,xs+s^2I-\sqrt{2A}\,yt+t^2I\right)^{-\nu}$ used to define a polynomial $C_{n,k}^{\nu}(x,y,A)$ in two variables x and y analogues to Gegenbauer matrix polynomials $C_{n}^{\nu}(x,A)$ in a single variable x can be expanded in powers of s and t in new ways, thus yielding additional results. For instance

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C_{n,k}^{\nu}(x,y,A)s^nt^k=\left(I-\sqrt{2A}\,xs+s^2I-\sqrt{2A}\,yt+t^2I\right)^{-\nu}$$

$$=\left[\left(I-\left(\sqrt{2A}/2I\right)xs-\left(\sqrt{2A}/2I\right)yt\right)^2-s^2\left(\left(A/2I\right)x^2-I\right)-t^2\left(\left(A/2I\right)y^2-I\right)-\left(A/I\right)xyst\right]^{-\nu}$$

$$=\left(I-\left(\sqrt{2A}/2I\right)xs-\left(\sqrt{2A}/2I\right)yt\right)^2\left[1-\frac{s^2\left(\left(A/2I\right)x^2-I\right)}{\left(I-\left(\sqrt{2A}/2I\right)xs-\left(\sqrt{2A}/2I\right)yt\right)^2}\right.$$

$$-\frac{t^2\left((A/2I)y^2-I\right)}{\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^2}-\frac{(A/I)xyst}{\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^2}\Bigg]^{-\nu}$$

$$=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\nu)_{j+p+r}\, s^{2j}\left((A/2I)x^2-I\right)^j t^{2j}\left((A/2I)y^2-I\right)^p\left((A/I)xyst\right)^r}{j!\,p!\,r!\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^{2j+2p+2r+2\nu}}.$$

By applying the relation (9) and (10), after little simplification and then equating the coefficients $s^n t^k$, we obtain

$$C_{n,k}^{\nu}(x,y,A)=\sum_{j=0}^{[n/2]}\sum_{p=0}^{[k/2]}\sum_{r=0}^{\min(n,k)}\frac{(2\nu)_{n+k}\left((A/2I)x^2-I\right)^j\left((A/2I)y^2-I\right)^p\left((\sqrt{2A}/2I)x\right)^{n-2j}\left((\sqrt{2A}/2I)y\right)^{k-2p}}{2^{2p+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!\,(n-r-2j)!\,(k-r-2p)!}. \quad (75)$$

Remark. 6.1. Let us now employ (75) to discover new double generating matrix function for $C_{n,k}^{\nu}(x,y,A)$ consider the double sum

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\beta I)_{n+k}\,C_{n,k}^{\nu}(x,y,A)\,s^n t^k}{(2\nu)_{n+k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[n/2]}\sum_{p=0}^{[k/2]}\sum_{r=0}^{\min(n,k)}\frac{(\beta I)_{n+k}\left((A/2I)x^2-I\right)^j\left((A/2I)y^2-I\right)^p\left((\sqrt{2A}/2I)x\right)^{n-2j}\left((\sqrt{2A}/2I)y\right)^{k-2p}s^n t^k}{2^{2p+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r} j!\,p!\,r!(n-r-2j)!(k-r-2p)!},$$

where βI and A are matrices in $C^{N\times N}$. Using the relation (12) and (9) would yield

$$=\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^{-\beta I}\sum_{j,p,r=0}^{\infty}\frac{\left(\frac{\beta I}{2}\right)_{j+p+r}\left(\frac{\beta I}{2}+\frac{I}{2}\right)_{j+p+r}}{\left(\frac{2\nu+1}{2}\right)_{j+p+r}\text{j!p!r!}}\left\{\frac{\left((\text{A/2I})\text{x}^2-I\right)s^2}{\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^2}\right\}^j$$

$$\times\left\{\frac{\left((\text{A/2I})\text{y}^2-I\right)t^2}{\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^2}\right\}^p\left\{\frac{(\text{A/I})xyst}{\left(I-\left(\sqrt{2A}/2I\right)x\,s-\left(\sqrt{2A}/2I\right)yt\right)^2}\right\}^r,$$

we obtain the following double generating matrix functions

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\beta I)_{n+k}\,C_{n,k}^{\nu}(x,y,A)\,s^n t^k}{(2\nu)_{n+k}}$$

$$=\rho^{-\beta I}\ \text{F}^{(3)}\left[\begin{matrix}\frac{\beta}{2}I,\frac{\beta}{2}I+\frac{I}{2}::-;-;-:-;-;-; \\ \qquad\qquad\frac{\left((\text{A/2I})\text{x}^2-I\right)s^2}{\rho^2},\ \frac{\left((\text{A/2I})\text{y}^2-I\right)t^2}{\rho^2},\frac{(\text{A/I})xyst}{\rho^2} \\ \frac{2\nu+I}{2}::\ -;-;-:-;-;-;\end{matrix}\right], \qquad (76)$$

where $\rho=\left(I-\left(\sqrt{2A}/2I\right)xs-\left(\sqrt{2A}/2I\right)yt\right)$ and $F^{(3)}[x,y,z]$ denotes a general triple hypergeometric series defined by $[23]$. In particular, if βI is matrix unity in $C^{N\times N}$ then (76) degenerates into the generating relation used to define $C_{n,k}^{\nu}(x,y,A)$.

Remark. 6.2. Let us now return to (75) and consider the double sum

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{C_{n,k}^{\nu}(x,y,A)s^n t^k}{(2\nu)_{n+k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[n/2]}\sum_{p=0}^{[k/2]}\sum_{r=0}^{\min(n,k)}\frac{\left((\mathrm{A/2I})\mathrm{x}^2-I\right)^j\left((\mathrm{A/2I})\mathrm{y}^2-I\right)^p\left((\sqrt{2\mathrm{A}}/2\mathrm{I})\,x\right)^{n-2j}\left((\sqrt{2\mathrm{A}}/2\mathrm{I})\,y\right)^{k-2p}s^n t^k}{2^{2p+2p+r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}\mathrm{j!\;p\;!\;r\;!}\;(\mathrm{n-r-2j})!\;(k-\mathrm{r-2p})!}$$

$$=\sum_{j,p,r=0}^{\infty}\frac{\left((\mathrm{A/2I})\mathrm{x}^2-I\right)^j\left((\mathrm{A/2I})\mathrm{y}^2-I\right)^p\left((A/2I)xyst\right)^r s^{2j}t^{2p}}{2^{2p+2p+2r}\left(\frac{2\nu+1}{2}\right)_{j+p+r}\mathrm{j!\;p\;!\;r\;!}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left((\sqrt{2\mathrm{A}}/2\mathrm{I})\,x\,s\right)^n\left((\sqrt{2\mathrm{A}}/2\mathrm{I})\,y\,t\right)^k}{n!\;k!}$$

$$=e^{(\sqrt{2A}/2I)(x\,s+y\,t)}\sum_{j,p,r=0}^{\infty}\frac{1}{\left(\frac{2\nu+1}{2}\right)_{j+p+r}\mathrm{j!p!r!}}\left\{\frac{\left((\mathrm{A/2I})\mathrm{x}^2-I\right)s^2}{4}\right\}^j\left\{\frac{\left((\mathrm{A/2I})\mathrm{x}^2-I\right)t^2}{4}\right\}^j\left\{\frac{(\mathrm{A/2I})xyst}{2}\right\}^r$$

$$=e^{(\sqrt{2A}/2I)(x\,s+y\,t)}\mathrm{F}^{(3)}\left[\begin{array}{c}-::-;-;-:-;-;-;\\ \frac{\left((\mathrm{A/2I})\mathrm{x}^2-I\right)s^2}{4},\frac{\left((\mathrm{A/2I})\mathrm{y}^2-I\right)t^2}{4},\frac{(A/2I)\,xyst}{2}\\ \frac{2\nu+I}{2}::-;-;-:-;-;-;\end{array}\right]. \qquad (77)$$

Remark 6.3. It is easy to drive the following property of $C^{\nu}_{n,k}(x,y,A)$. Relation (20) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{\nu}_{n,k}\left(x,y,A\right)s^n t^k=\mu^{-1},\quad \text{where}\quad \mu=\left(I-\sqrt{2A}\,x\,s+s^2I-\sqrt{2A}\,yt+t^2I\right)^{\nu}. \qquad (78)$$

Now replacing x by $\frac{x-\left(2I/\sqrt{2A}\right)s}{\mu}$, y by $\frac{y-\left(2I/\sqrt{2A}\right)t}{\mu}$, s by $\frac{u\,I}{\mu}$ and t by $\frac{\nu I}{\mu}$ in (20), after little mathematical simplification, considering that $\mu^2 I=\mu^2$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C^{\nu}_{n,k}\left(\frac{x-\left(2I/\sqrt{2A}\right)s}{\mu},\frac{y-\left(2I/\sqrt{2A}\right)t}{\mu},A\right)\mu^{-n-k-2\nu}u^n v^k$$

$$=\left[I-\sqrt{2A}\,x\,(s+u)+(s+u)^2 I-\sqrt{2A}\,y\,(t+v)+(t+v)^2 I\right]^{-\nu},$$

so that from (78), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}\left(\frac{x-\left(2I/\sqrt{2A}\right)s}{\mu},\frac{y-\left(2I/\sqrt{2A}\right)t}{\mu},A\right)\mu^{-n-k-2\nu}u^n v^k=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}\left(x,y,A\right)(s+u)^n(t+v)^k$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\sum_{j=o}^{k}\frac{n!k!\,C_{n,k}^{\nu}\left(x,y,A\right)u^{n-r}v^{k-j}s^r t^j}{\mathrm{r!\,j!(n-r)!\,(k-}j)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=o}^{\infty}\frac{(n+r)!(k+j)!\,C_{n+r,k+j}^{\nu}\left(x,y,A\right)u^n v^k\,s^r t^j}{\mathrm{r!\,j!}\,n!k!},$$

equating the coefficients of $u^n v^k$ form above we get

$$\mu^{-n-k-2\nu}C_{n,k}^{\nu}\left(\frac{x-\left(2I/\sqrt{2A}\right)s}{\mu},\frac{y-\left(2I/\sqrt{2A}\right)t}{\mu},A\right)=\sum_{r=0}^{\infty}\sum_{j=o}^{\infty}\frac{(n+r)!(k+j)!\,C_{n+r,k+j}^{\nu}\left(x,y,A\right)s^r t^j}{\mathrm{r!\,j!}\,n!k!}. \tag{79}$$

Remark 6.4. One can easily introduce a bilinear double generating matrix function as an example of the equation (79), by replacing x by $\frac{x-\left(2I/\sqrt{2A}\right)s}{\mu}$, y by $\frac{y-\left(2I/\sqrt{2A}\right)t}{\mu}$, s by $-\frac{su}{\mu}$ and t by $-\frac{tv}{\mu}$ in (77), with multiplying the relation (77) by $\mu^{-2\nu}$ where $\mu=\left(I-\sqrt{2A}\,x\,s+s^2 I-\sqrt{2A}\,yt+t^2 I\right)^{\nu}$ and with the aid of (11) with some mathematical simplification, we obtain

$$\mu^{-2\nu}\,e^{\left(\frac{\sqrt{A}}{2I}\right)\left(-\frac{\left(x-\frac{2I}{\sqrt{2A}}\right)su+\left(y-\frac{2I}{\sqrt{2A}}\right)tv}{\mu^2}\right)}\,\mathrm{F}^{(3)}\left[\begin{array}{l} -::-;-;-:-;-;-; \\ \qquad\qquad\qquad\qquad \frac{\eta s^2u^2}{4\mu^4},\frac{\zeta t^2v^2}{4\mu^4},\frac{(A/2I)\lambda sutv}{2\mu^4} \\ \frac{2\nu+I}{2}::-;-;-:-;-;-; \end{array}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=o}^{\infty} \Phi_2 [-n,-k;2\nu;u,v] C_{n,k}^{\nu}(x,y,A) s^n t^k . \tag{80}$$

Where Φ_2 is one the seven confluent forms of the four Appell series defined by Humbert [24] for convenience, we let

$$\eta = \left((\mathrm{A}/(2\mathrm{I}))x^2 - I + \left(\sqrt{2A}/2I\right) y - t^2 I \right)$$

$$\zeta = \left((\mathrm{A}/(2\mathrm{I}))\mathrm{y}^2 - I + \left(\sqrt{2A}/2I\right) x - s^2 I \right)$$

$$\lambda = \left(x - \left(2I/\sqrt{2A}\right)s \right)\left(y - \left(2I/\sqrt{2A}\right)t \right).$$

7. Expand the Gegenbauer matrix polynomials of two variables in series of $H_n(x,y,A)$:

Let us now employ (21) and (17) with considering that each matrix commutes with itself. From (21), one gets

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}(x,y,A) t^{n+k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{j=0}^{[k/2]} \frac{(-1)^{r+j} (\nu)_{n+k-r-j} \left(\sqrt{2A}\,x\right)^{n-2r} \left(\sqrt{2A}\,y\right)^{k-2r} t^{n+k}}{r!\,j!\,(n-2r)!\,(k-2j)!},$$

now using (12)

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{r+j} (\nu)_{n+k+r+j} \left(\sqrt{2A}\,x\right)^{n} \left(\sqrt{2A}\,y\right)^{k} t^{n+k+2r+2j}}{r!\,j!\,n!\,k!},$$

from (17) it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n,k}^{\nu}(x,y,A) t^{n+k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{[n/2]} \frac{(-1)^{r+j} (\nu)_{n+k+r+j}\, y^s \left(\sqrt{2A}\,y\right)^{k} t^{n+k+2r+2j} H_{n-2s}(x,y,A)}{r!\,j!\,k!\,(n-2s)!\,s!},$$

by applying the relation (12), (11) and (5) respectively, after little simplification the above relation reduce to the following form

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{r+j}(\nu)_{n+k+r+j}\left(\sqrt{2A}\,y\right)^{k}}{n!k!r!j!}\sum_{s=0}^{k}\frac{\left[\nu+(n+k+r+j)I\right]_{S}(-k)_{S}}{(-1)^{s}s!}$$
$$\times\left(\frac{t}{\sqrt{2A}y}\right)^{s}(A)_{n+k+r+j}H_{n}(x,y,A)t^{n+k+2r+2j}.$$

Thus, above relation can be rewritten as

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{r+j}\left(\sqrt{2A}\,y\right)^{k}}{n!k!r!j!}\;{}_{2}F_{0}\left[-k\,I,\nu+(n+k+r+j)I;-;\left(-\frac{1}{\sqrt{2A}y}t\right)\right]$$
$$\times(\nu)_{n+k+r+j}H_{n}(x,y,A)t^{n+k+2r+2j},$$

again from (12), one gets

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{j=0}^{[k/2]}\frac{(-1)^{r+j}\left(\sqrt{2A}\,y\right)^{k-2j}}{(n-2r)!(k-2j)!r!j!}\;{}_{2}F_{0}\left[-k\,I,\nu+(n+k+r+j)I;-;\left(-\frac{1}{\sqrt{2A}y}t\right)\right]$$
$$\times(\nu)_{n+k-r-j}H_{n}(x,y,A)t^{n+k},$$

now, equating the coefficient of t^{n+k} we obtain the following expansion of $C_{n,k}^{\nu}(x,y,A)$ as series of two-variable Hermite matrix polynomials in the form

$$C_{n,k}^{\nu}(x,y,A)\quad=\sum_{r=0}^{[n/2]}\sum_{j=0}^{[k/2]}\frac{(-1)^{r+j}\left(\sqrt{2A}\,y\right)^{k-2j}}{!r!j!(n-2r)!(k-2j)!}$$
$$\times\;{}_{2}F_{0}\left[-kI,\nu+(n+k+r+j)I;-;\left(-\frac{t}{\sqrt{2A}y}\right)\right](\nu)_{n+k-r-j}H_{n}(x,y,A).\qquad(81)$$

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