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S Squeezing And Correlation Of Pure Spin 3/2 System

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الملخص

الانضغاطية والارتباط للحالة المغزلية النقية (3/2)

ان دراسة عملية الانضغاط للحالة المغزلية تتطلب اختزال مبدأ اللايقين لمركبات المتجه $\vec{S}$ لقيمة محددة.

تم دراسة هذه الظاهرة بواسطة الباحثين: كتاجاوا، أودي و مالاش للحالة المغزلية النقية ذو الرتبة (1) لمتغير واحد، وكانت القيمة العظمى للانضغاط =1 .

قمنا في هذا البحث بدراسة الحالة المغزلية ذو الرتبة الاعلى (3/2) باستخدام ثلاث أنظمة ذو الرتبة (1/2)، ثم دراسة عملية الانضغاط و الارتباط بين كل حالتين مغزليتين، وذلك باستخدام شرط العالم لاكين.

من خلال البحث تبين أن شرط حدوث الانضغاط، وجود متغيرين اثنين (γ ، δ) وقيمتين عظميين للانضغاط والارتباط (أعلى قيمة للانضغاط = 1.292). كما تبين انه كلما زادت قيمة الارتباط فإن قيمة الانضغاط تزداد، وان من شروط الانضغاط وجود الترابط المغزلي بين مستويات الجسم.

Abstract

The notion of spin squeezing involves a reduction in the uncertainty of a components of spin vector $\vec{S}$ below a certain limit. This aspect has been studied earlier by Kitagawa, Udea [1], Mallesh [2], Jian Ma, Xiaoguang Wang [3] and Keyu Xia, Jason Twamley [4] for spin state. In this paper we extended to the higher spin 3/2 system, using three spin 1/2 system and study it's squeezing behavior. A discussion of spin-spin correlations and their link with squeezing will be studied. It will be shown that the existence of correlations is only a necessary condition for squeezing to be present.

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1- Introduction

The concept of squeezing is associated with quantum fluctuations or uncertainties in non-commuting observables of a quantum system.

The notion of squeezing was first introduced in the case of harmonic oscillator [5] and subsequently it has been extensively studied in Bosonic system [2,6].

The concept of spin is fascinating in quantum theory defined through the commutation relations: $(\hbar = 1)$, [13]

$$\vec{j} \times \vec{j} = i\,\vec{j} \qquad (1)$$

Which are common to intrinsic spin operator $\vec{s}$ consisting of three Hermitian components S_x , S_y and S_z. These operators are postulated according to equation (1). The square of the spin operator $\vec{s}$ defined as:

$$S^2 = \vec{S} \cdot \vec{S} = S_x^2 + S_y^2 + S_z^2 \qquad (2)$$

and

$$[S_x\,,\,S_y] = i\,S_z \qquad (x, y, z \text{ cyclic}) \qquad (3)$$

The uncertainty relationship for these spin operators is given by:

$$\Delta S_x^2\,\Delta S_y^2 \geq \frac{\langle S_z \rangle^2}{4} \qquad (4)$$

For discussing the squeezing nature of spin system, we restrict ourselves to Heisenberg's relationship that a spin state could be regarded as squeezed if:

$$\Delta S_x^2 < \frac{|\langle S_z \rangle|}{2} \qquad (5)$$

or

$$\Delta S_y^2 < \frac{|\langle S_z \rangle|}{2} \qquad (6)$$

A more stringent condition which have been advocated by Wineland [5].

$$\zeta = \left[\frac{2S(\Delta S_\perp)^2}{\left|\langle \psi | \vec{S} \cdot \vec{V} | \psi \rangle\right|^2}\right]^{\frac{1}{2}} < 1 \qquad (7)$$

Kitagawa and Ueda [1] have argued that it would be possible to cancel out fluctuation in one direction normal to $\vec{s}$ at the expense of the these provided quantum correlations are established among the elementary spinors which constitute the spin state. The spin stat are divided into two classes for pure

spin state $|\psi\rangle$ referred to oriented and non-oriented has been suggested [2,9]. It has been shown that all oriented spin state are not squeezed, and the squeezing can be present in non-oriented spin state.
The present paper which addresses these intrinsic notions is organized as follows.
In section 2, we construct the pure spin 3/2 state and the squeezing conditions will be studied (Eq. 5, 6) based on the uncertainty relation (Eq. 4), it will be shown that the squeezing is exhibited by only non-oriented system, therefore we shall discuss the squeezing behavior of pure spin 3/2 system.
In section 3 we shall look into spin - spin correlation, which exist between these three spinors when they combined to yield a spin 3/2 state. And we shall show that the necessary conditions for the squeezing to be present the existence of spin - spin correlations.

2- Squeezing for pure spin 3/2 state

A normalized pure state of spin 3/2 system can be expressed in term of angular momentum state: $\left|\frac{3}{2}\ \frac{3}{2}\right\rangle, \left|\frac{3}{2}\ \frac{1}{2}\right\rangle, \left|\frac{3}{2}\ \frac{-1}{2}\right\rangle\ and\ \left|\frac{3}{2}\ \frac{-3}{2}\right\rangle$ with respect to the *z*- axis in the frame *xyz* as:

$$|\psi\rangle = a_1\left|\frac{3}{2}\ \frac{3}{2}\right\rangle + a_2\left|\frac{3}{2}\ \frac{1}{2}\right\rangle + a_3\left|\frac{3}{2}\ \frac{-1}{2}\right\rangle + a_4\left|\frac{3}{2}\ \frac{-3}{2}\right\rangle \tag{8}$$

Where a_1, a_2, a_3 and a_4 are complex numbers, which satisfying the condition:

$$|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2 = 1 \tag{9}$$

The spin operators S_x , S_y and S_z for pure spin 3/2 are given by, [10]

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \tag{10}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix} \tag{11}$$

and

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \qquad (12)$$

The expectation values $\langle S_x\rangle$, $\langle S_y\rangle$ *and* $\langle S_z\rangle$ for the state refer to equation (9) are:

$$\langle S_x\rangle = \frac{1}{2}[\sqrt{3}a_1^*a_{2+}a_2^*(\sqrt{3}a_1 + 2a_3) + a_3^*(2a_2 + \sqrt{3}a_4) + \sqrt{3}a_3a_4^*] \qquad (13)$$

$$\langle S_y\rangle = \frac{i}{2}[-\sqrt{3}a_1^*a_{2+}a_2^*(\sqrt{3}a_1 - 2a_3) + a_3^*(2a_2 - \sqrt{3}a_4) + \sqrt{3}a_3a_4^*] \qquad (14)$$

$$\langle S_z\rangle = \frac{1}{2}[3\,|a_1|^2 + |a_2|^2 - |a_3|^2 - 3|a_4|^2] \qquad (15)$$

So that these defined the mean spin vector $\langle\vec{S}\rangle$, through:

$$\langle\vec{S}\rangle = \langle S_x\rangle_{\hat{i}} + \langle S_y\rangle_{\hat{j}} + \langle S_z\rangle_{\hat{k}} \qquad (16)$$

Now the polar angles α, β of mean spin direction [2] with respect to *xyz* are given by:

$$\alpha = tan^{-1}\left(\frac{\sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2}}{\langle S_z\rangle}\right) \qquad (17)$$

and

$$\beta = tan^{-1}\ \frac{\langle S_y\rangle}{\langle S_x\rangle} \qquad (18)$$

Now the number of parameters which define the state $|\psi\rangle$ (Eq. 8) are 8 parameters, these parameters reduced by two, one for a normalization condition and the other for over phase.

To reduce the number of parameters, applying the rotation about *z* and *y* axes, we can transform *xyz* to *x'y'z'*, so that the number of parameters reduced by 2 .

It's clear from equations (13, 14) that $\langle S_{x'}\rangle$ *and* $\langle S_{y'}\rangle$ are not satisfied in the Lakin frame in *x'y'z'* [11].

For more condition to be satisfied it should be that $a_2' = a_2\,\langle S_z\rangle^2$ where the prime (′) refer to the state after rotation. Therefore, the new axis which have to be the Lakin frame $x_0y_0z_0$

$$\langle S_{x_0}\rangle = \langle S_{y_0}\rangle = 0 \qquad (19)$$

Under this condition, the number of parameters reduced by 4 .

Now, the pure spin state 3/2 in term of the angular momentum basis after these rotation can be written in the form:

$$|\psi\rangle = \begin{pmatrix} sin\delta \\ 0 \\ cos\delta\, e^{i\gamma} \\ 0 \end{pmatrix} \qquad (20)$$

It's clear from equation (20) that for wide range of δ the state $|\psi\rangle$ is obviously non-oriented.

For such non- oriented state which referred to the frame $x_0y_0z_0$ [2], the relevant quantities need for studying the squeezing (Eq's 5 , 6) turn out to be:

$$\Delta S_{x_0}^2 = \frac{1}{4}\left[3 + 4cos^2\delta + 4\sqrt{3}\, sin\delta\, cos\delta\, cos\gamma\right] \qquad (21)$$

$$\Delta S_{y_0}^2 = \frac{1}{4}\left[3 + 4cos^2\delta - 4\sqrt{3}\, sin\delta\, cos\delta\, cos\gamma\right] \qquad (22)$$

and

$$\langle S_{z_0}\rangle = \frac{1}{2}\left[3 - 4cos^2\delta\right] \qquad (23)$$

So that the squeezing conditions for S_{x_0} and S_{y_0} are respectively given by:

$$3 + 4cos^2\delta + 4\sqrt{3}\, sin\delta\, cos\delta\, cos_{\gamma} < |3 - 4cos^2\delta| \qquad (24)$$

or

$$3 + 4cos^2\delta - 4\sqrt{3}\, sin\delta\, cos\delta\, cos\gamma < |3 - 4cos^2\delta| \qquad (25)$$

It's clear from equation's (24, 25) that the state $|\psi\rangle$ is squeezing state for wide range of δ and γ as shown in the figure (1) and (2).

The absence of squeezing $0 < \delta < 49.5^\circ$ and $\gamma = 180^\circ$ is referred to other construction state. The figure (1) and (2) also shown that: at $\gamma = 180^\circ$ the maximum squeezing occurs at $\delta = 69.53^\circ$ and the value of squeezing $Q_x = Q_y = 1.292$

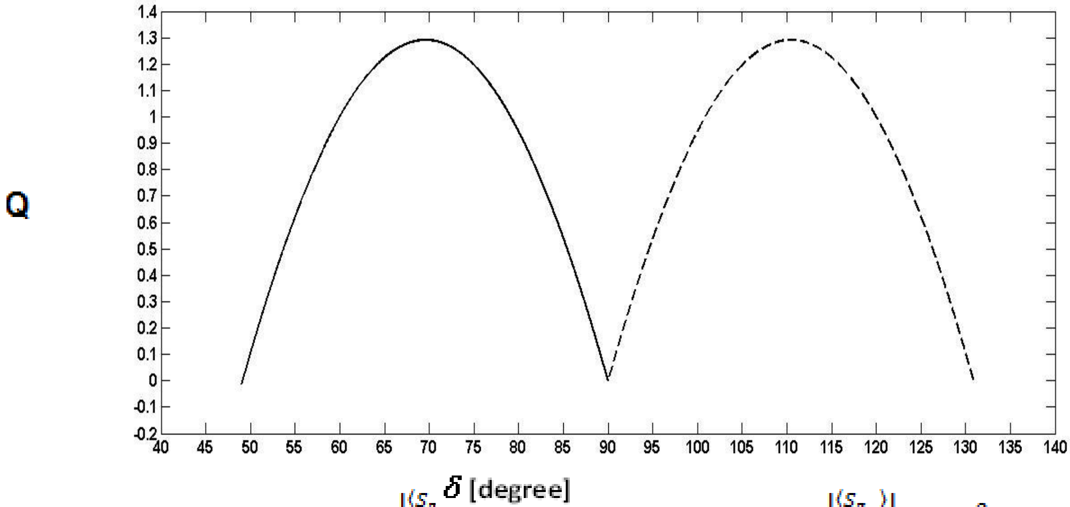

Figure (1): Variation of $\left|\frac{\langle S_{z_0}\rangle}{2}\right| - \Delta S_{x_0}^2$ (solid carve) & $\left|\frac{\langle S_{z_0}\rangle}{2}\right| - \Delta S_{y_0}^2$ (dashed carve) with respect to δ and γ =180°. Positive values of function imply present of squeezing.

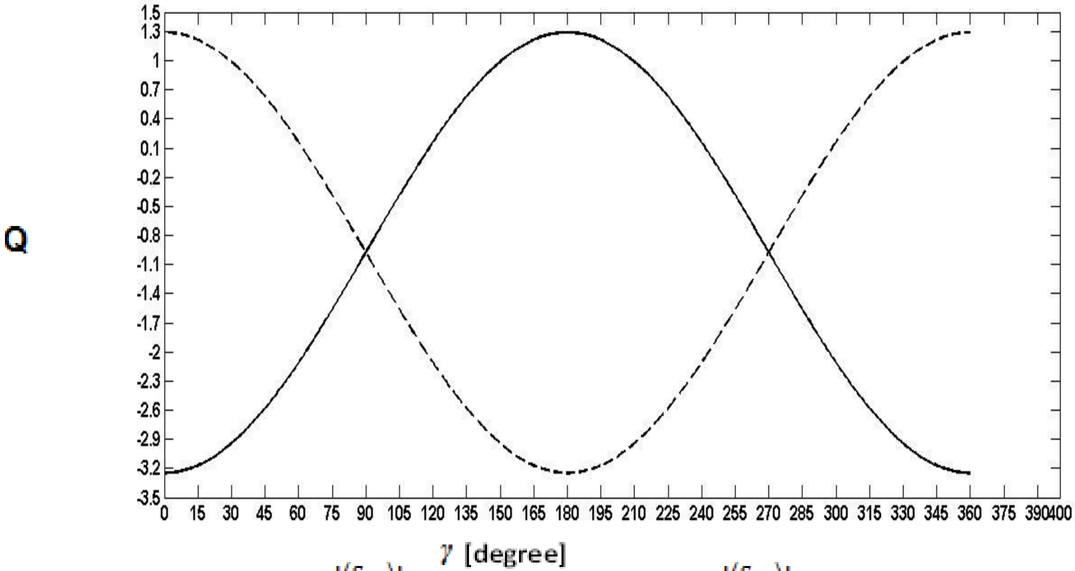

Figure (2): Variation of $\left|\frac{\langle S_{z_0}\rangle}{2}\right| - \Delta S_{x_0}^2$ (solid carve) & $\left|\frac{\langle S_{z_0}\rangle}{2}\right| - \Delta S_{y_0}^2$ (dashed carve) with respect to γ and δ =69.5°. Positive values of function imply present of squeezing.

3- Quantum Correlations

We have looked into the squeezing of pure spin 3/2 system which is consisting of three spinors, that squeezing in pure spin state system arises from the existence of quantum correlation. This can be done by employing the model in which a spin-*S* state is constructed using 3*S* spin 1/2 state. Indeed, such construction using spinors were known in mathematics and defined in their most general form by Cartan [2,12].

For three spin 1/2 system, there are several set of commuting operators which may be diagonalzed simultaneously [13]. Thus, the six operators for the 3 spinors *S* which may be diagonalized are: $S_1^2 \;\; S_2^2 \;\; S_3^2 \;\; S_1 \;\; S_2 \;\; S_3$

To generalize this realization by taking 3*S* 'up' spinors $u(\theta_l\ ,\ \varphi_l\ )$ [2]; l=1, …25

Where the K^{th} spinors is specified with respect to an axis of quantization $Q_K\,(\theta_K\ ,\ \varphi_K)$ in physical space.

Coupling three spinors lead to [13]:

$$|\,\psi\,\rangle = |u(\theta_1,\varphi_1)\ u\ (\theta_2,\varphi_2)\ u\ (\theta_3,\varphi_3)\rangle$$

$$= N_1 \sum_{m_1 m_2 m_3} D^{\frac{1}{2}}_{m_1\frac{1}{2}}\,(\theta_1\ \varphi_1\ 0)\ D^{\frac{1}{2}}_{m_2\frac{1}{2}}\,(\theta_2\ \varphi_2\ 0)\,D^{\frac{1}{2}}_{m_3\frac{1}{2}}\,(\theta_3\ \varphi_3\ 0)$$

$$C\left(\tfrac{1}{2}\ \tfrac{1}{2}\ J',\ m_1\ m_2\ m'\right)C\left(J'\ \tfrac{1}{2}\ J,\ m' m_3\ m\right)|\ Jm\rangle \qquad (26)$$

For such state the construction can be carried into three spinors with respect to quantization axes $Q_1(\theta_1\ ,\ \varphi_1)$, $Q_2(\theta_2\ ,\ \varphi_2)$, and $Q_3(\theta_3\ ,\ \varphi_3)$ in the frame *xyz*. Here the basis states are referred to the *z*- axis of *xyz* frame.

Now to construct the state that satisfy the squeezing condition itself, we will go to the frame $x_0y_0z_0$ where $\langle S_{x_0}\rangle = \langle S_{y_0}\rangle = 0$. This state $|\,\psi\,\rangle$ expressed in term of the spinors state with respect to z_0 in frame $x_0y_0z_0$ can be written in the matrix form

$$|\,\psi\,\rangle = \frac{\sqrt{3}\ i}{cos\ (\frac{\theta}{2})\sqrt{3cos^4(\frac{\theta}{2}) + 4\ sin^4(\frac{\theta}{2})}} \begin{pmatrix} cos^3\,(\frac{\theta}{2}) \\ 0 \\ \frac{2}{\sqrt{3}}\,sin^2(\frac{\theta}{2})\ \ cos(\frac{\theta}{2}) \\ 0 \end{pmatrix} \qquad (27)$$

We now establish explicitly for *S*=3/2, the connection between squeezing and spin - spin correlation that exist between the constituent spinors.

Any spin 3/2 state constructed by using 3 spinors is said to be constructed if the correlation matrix C^{ij} defined through the elements [2]:

$$C^{ij}_{\mu\nu} = \langle S_{i\mu}\, S_{j\nu}\rangle - \langle S_{i\mu}\rangle\langle S_{j\nu}\rangle \qquad (28)$$

$i\,, j = 1\,, 2\,, 3$

$\mu\,, \nu = x\,, y\,, z$

is non-zero. Here $S_{i\mu}$ and $S_{j\nu}$ are the spin components associated with the three spinors.

For the state $|\,\psi\,\rangle$ in equation (26), the correlation matrix is diagonal in the frame $x_0y_0z_0$ which are given by:

$$C^{ij}_{x_0x_0} = \frac{3\left[3cos^4\frac{\theta}{2} + 8sin^2\frac{\theta}{2}cos^2\frac{\theta}{2} + \frac{28}{3}sin^4\frac{\theta}{2}\right]}{4(3cos^4\frac{\theta}{2} + 4sin^4\frac{\theta}{2})} \qquad (29)$$

$$C^{ij}_{y_0y_0} = \frac{3\left[3cos^4\frac{\theta}{2} - 8sin^2\frac{\theta}{2}cos^2\frac{\theta}{2} + \frac{28}{3}sin^4\frac{\theta}{2}\right]}{4(3cos^4\frac{\theta}{2} + 4sin^4\frac{\theta}{2})} \qquad (30)$$

$$C^{ij}_{z_0z_0} = \frac{3\left[9cos^4\frac{\theta}{2} + \frac{4}{3}sin^4\frac{\theta}{2}\right]}{4(3cos^4\frac{\theta}{2} + 4sin^4\frac{\theta}{2})} \qquad (31)$$

The above expressions show that, for all values of θ, the eigenvalue of correlations satisfy [2] (Figure 3):

$$0 < C^{ij}_{\mu\nu} \le 2.25 \qquad (32)$$

Also, we can see that the trace of correlation matrix:

$$T_r\, C^{ij}_{\mu\nu} = \frac{15}{4} \qquad (33)$$

Thus, the spin spin matrix correlations is the sufficient conditions for a spin 3/2 system to be squeezed.

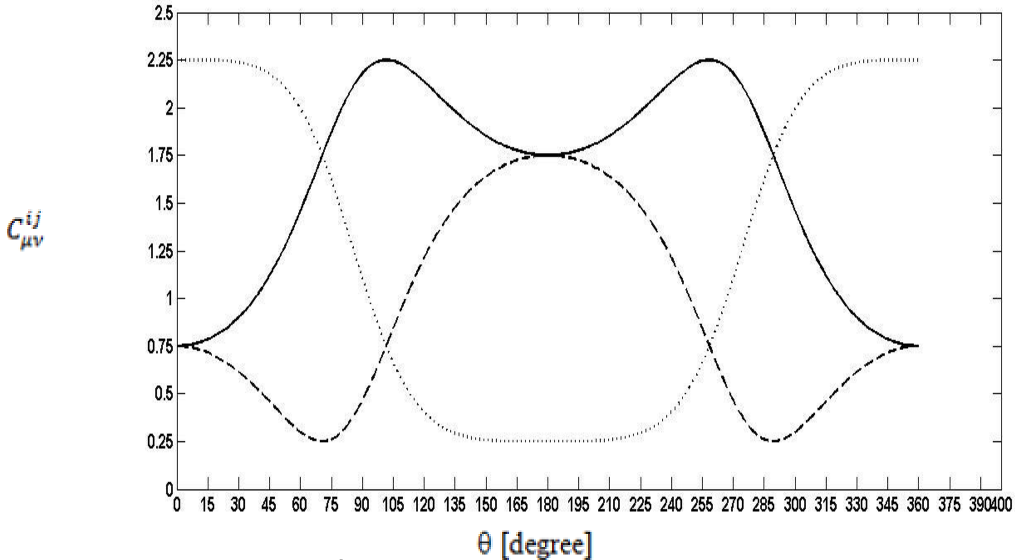

Figure (3): Variation of spin – spin correlation C_{xx} (solid carve), C_{yy} (dashed carve) and C_{zz} (dotting carve).

4- Summary

We have looked into the squeezing aspect of a pure bipartite state consisting of three spinors. A suitable state for *S*=3/2 construction has been obtained, which is satisfied the Lakin frame condition's $\langle S_{x_0}\rangle = \langle S_{y_0}\rangle = 0$.

A suitable criterion for squeezing of such state has been obtained which is a generalization of the squeezing criterion for state possess wide range of squeezing which depend on two variables δ and γ .

However, we have to mention here that for pure spin 1state [2], the squeeze was depend on one variable only and maximum value of squeezing is 1. Our study in this paper show that the squeezing was depend on two variables and the maximum squeezing is 1.292 as shown in figures (1) & (2).

While squeezing is established in the case of pure spin 3/2 system due to the self-correlation.

The existence of spin –spin correlation is only the sufficient condition for the state to be squeezed. Our studies are also shown that when the value of correlation is increase the value of squeezing also increase, this due to more spin–spin correlations.

Our study in this paper gives justification made by Kitagawa and Udea [1] regarding that the link between the squeezing and spin-spin correlations.

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